Macroeconomics (ECON205A)

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1 Introduction

In this course, we will cover the basics of productivity growth, consumption, investment and savings, fiscal policy and asset market equilibrium. We begin first with dynamic general equilibrium models, at the most simple form, and then expand on this model.

Firstly, it is important to gave a brief primer on the history of economic thought, and particularly how the macroeconomic paradigms which have dominated area a result of the evolution of the discipline, the varying degrees on mathematical intensity, and the important legacy figures who have contributed to the development of these models.

In our coverage of neoclassical macroeconomics we start with a very general overview of components of neoclassical growth models in **Section 2**. We then cover command economy models, where there is either one agent or one central planning agency which allocates scarce resources in the economy. These are the models in **Section 3**, including Robinson Crusoe (single agent), command economy, and Yeoman peasant models. These are autarkic (no trade) models with no distortions. Model extensions in the command economy introduce new elements, such as time variance and productivity shocks. The optimization problem remains the same. In **Section 4**, we look at dynamic stochastic general equilibrium (DSGE) models where there is a market that must clear, with relative pricing and trading. This includes models with taxation and durable goods produced by firms. The optimization problem therefore changes, as we no longer have a social planner, but instead we use a decentralized competitive equilibrium to ensure pareto optimality. And finally, in **Section 5** we cover one model on complete markets. It is an extended DSGE model that begins to account for asset prices.

2 Neoclassical growth model

2.1 Foundations

We begin our study of neoclassical growth models with two key models: The Solow Growth model and the Ramsey Model. The Solow growth model is discussed briefly to give the foundation, but it is not used for problem solving given it does not account for a utility maximization and assumes savings is exogenous. The Ramsey growth model is primarily the framework we will use moving forward. The Ramsey model allows us to model a particular growth pathway in the long run, explicitly based on agent behavior, where we find the path by optimizing consumption at the household level. Here we identify how the optimal level of output in an economy is determined, and how it is allocated between (1) consumption, c_t, and (2) capital accumulation, (or future consumption, c_{t+1})

At a very fundamental level, there are two approaches to optimize a Ramsey model:

1. Golden rule: where no discount rate is applied, but a steady state equilibrium is found for the single period. Here the solution is always *myopic and unstable*, and consumption in the long run is just equal to output. This requires a constant level of capital, k^*, and a disturbance to the pathway cannot be modeled as the optimal consumption becomes unachievable.

2. Optimal solution approach: where we maximize the sum of discounted utility across all periods, using dynamic programming or the Lagrange approach (sequential equilibrium strategy). An equilibrium can be found using the derived necessary condition, the euler equation, and the given resource constraint on capital. The static equilibrium, with no shocks, yields:
$$c_t = c^* \quad k_t = k^* \quad \triangle c_t = 0 \quad \triangle k_t = 0$$

2.2 Basic components of the model

Production function

$$y = A \cdot F(K, N)$$

such that F is the standard production capacity function, that takes capital and labor services as an input. A is a parameter which captures productivity, able to augment output. Y is total output. K as the stock of capital and N as the stock of labor are heterogeneous among agents. A few things about the production function:

- it is continuous and differentiable in the first order condition, we can assume the function is (strictly) concave
- $\dfrac{dF}{dK} \geq 0$

- $\frac{dF}{dN} \geq 0$ Note the simplest form of a production model would be where output is equal to $Y(A, K)$ such taht $Y_t = A_t F(k_t, n_t) = A_t f(k_t)$ where labor supply is perfectly inelastic, such that $n_t = 1$ for all t.

Equation of motion

We have an equation of motion of capital[1], such that:

$$\triangle K_{t+1} = AF(K_t, N_t) - \delta K_t - C_t$$

Where $\triangle K_{t+1} = K_{t+1} - K_t$. We set up this general equation as we are working in discrete time. This equation tells us how optimal savings occurs at the household level. Capital in the next period is a function of productive output, less the depreciation of capital and any consumption that occurs. We assume that households own the capital stock.

Utility function

Now for a household, we assume there is *one representative household* or agent, which has the same utility function. The utility function is a summation of the consumption of goods across all periods, $t = 1, ...T$. The standard utility function is:

$$U_o = \sum_{t=0}^{T} \beta^t u(c_t)$$

We make the following assumptions about the utility function:

- utility function is downward sloping and a convex curve
- both monotonic and continuous
- $\lim_{c_t \to \infty} \frac{dU}{dc_t} = \infty$
- marginal utility of each c_t is positive for all $c_t \geq 0$
- additively separable over time

Euler equation

The Euler equation is a fundamental result from the Ramsey growth model, that tells us how the capital stock changes between different time periods, t. In the two period model, it tells us how capital changes from period 0 to 1. Most often, it is written as:

$$u'(c_t) = \beta u'(c_{t+1}) \cdot (1 - \delta + f'(k_{t+1}))$$

3 Command economy

We start with *command economy* models where the government, or the social planner, makes all major economic decisions. Command economy models are characterized by achieving socially optimal outcomes without distortions, as the social planner chooses production and resource allocation. This framework avoids the needs for markets to clear, and eliminates all supply and demand determinations for goods. We define the **command optimum** as the optimal allocation of resources over the technologically possible (productively feasible) frontier. In a command economy, at the optimum, we have a **Pareto optimum** where goods cannot be redistributed to make anyone better off without making someone else worse off. The command optimum is a unique value.

The *necessary conditions* that need to be satisfied to reach a command optimum are the following:

1. The euler equation (see 2.2), which states that the marginal rate of substitution of consumption between two periods is equivalent to the marginal productivity of capital in the second period. This is the most simplified version of the euler equation.

[1]The equation of motion of capital is also referred to in the notes as the capital accumulation equation, and the resource identity. It is *not* the resource constraint. While it seems confusing, the reason for the different names is such that it shows the relationship between capital stock with other variables, primarily consumption (and later on, investment) that governs the motion of capital across periods.

2. The capital accumulation function must hold for all time periods, where the productivity of capital plus the remaining capital, less depreciation, should equal consumption and the capital in the next period

3. All capital must be exhausted. This implies that k_T (or k_2 in the two period model) will be zero such that all available resources are exhausted.

Given the necessary conditions, we need to show they are also sufficient. This requires concavity, which we already assumed, and compactness. So, for the set of allocations, (c_0, c_1, k_1, k_2) we must have that they satisfy the inequality constraints initially given, and are closed and bounded as a subset of $\mathbb{R}^4_+$.

3.1 Two-period model

In two periods, $t = 0, 1$, we have the following model components:

- Utility function: $U = u(c_0, c_1)$
- Capital accumulation function: $k_1 = f(k_0) + (1-\delta)k_0 - c_0$
- k_0, k_1 are both determined during period 0
- c_0, c_1 are chosen to maximize utility over consumption pairs
- The model is subject to the following *constraints*:
  - $c_0, c_1, k_1, k_2 \geq 0$
  - $f(k_0) + (1-\delta)k_0 \geq k_1 + c_0$
  - $f(k_1) + (1-\delta)k_1 \geq k_2 + c_1$
- State variable: k_t for $t = 1, 2$; control variable: c_0 for $t = 0, 1$
- We assume $f(k) \geq 0$ for all $k \geq 0$

With all the model components, we now set up a Lagrange equation, where there is a Lagrange multiplier attached to all the constraints. For completeness we include multipliers on the nonzero constraints, but normally we can ignore these, and assume the constraints are binding. The Lagrange set up[2] is:

$$\mathcal{L} = u(c_0, c_1) + \lambda_0(f(k_0) + (1-\delta)k_0 - k_1 - c_0) + \lambda_1(f(k_1) + (1-\delta)k_1 - k_2 - c_1) + [\mu_0 c_0 + \mu_1 c_1 + v_1 k_1 + v_2 k_2]$$

with FOCs:

$$\begin{aligned}
\frac{\partial L}{\partial c_o} &= \frac{\partial U}{\partial c_0} - \lambda_0 + \mu_0 = 0 \\
\frac{\partial L}{\partial c_1} &= \frac{\partial U}{\partial c_1} - \lambda_1 + \mu_1 = 0 \\
\frac{\partial L}{\partial k_1} &= 0 \implies \lambda_0 = \lambda_1[f'(k_1) + 1 - \delta] + v_1 \\
\frac{\partial L}{\partial k_2} &= 0 \implies -\lambda_1 + v_2 = 0
\end{aligned}$$

We also know that the utility function is twice continuously differentiable, and the limit on the marginal utilities is infinity (as the consumption goods tends to zero). Similarly, the marginal productivity of capital also has a limit to infinity, such that:

$$\lim_{k \to 0} f'(k) = \infty$$

With all of this, we also look at the conditions that must be satisfied to get solutions for c_0, c_1, k_1. We must have:

$$c_0 > 0 \quad c_1 > 0 \quad k_1 > 0 \qquad \mu_0 = \mu_1 = v_1 = 0$$

[2] Note that in most of problem solving, mechanically we ignore the non-negativity constraints on the control variables, which are explicitly included below. This is because we, in the future, impose onto these conditions complementary slackness, and as such, we know they will bind (=0). For the purpose of giving as much information as possible in the theory section, we include them.

Rearranging the FOCs from above we get the necessary conditions:

$$(1) \quad \lambda_1 = v_2 \quad \text{and} \quad (2) \quad \frac{\partial u}{\partial c_1} = \lambda_1 > 0$$

and the necessary conditions for the household's optimum:

$$(3) \quad \frac{\partial u}{\partial c_0} = \lambda_0 \qquad (4) \quad \frac{\partial u}{\partial c_1} = \lambda_1 \qquad (5) \quad \lambda_0 = \lambda_1[f'(k_1) + (1-\delta)]$$

Where we can then plug in the marginal utilities of consumption in period $0, 1$ for the respective λ_0, λ_1, to get the *Euler condition* (see 2.2):

$$\frac{\partial u}{\partial c_0} = \frac{\partial u}{\partial c_1} \cdot [f'(k_1) + (1-\delta)]$$

Now, with the Euler equation, we have the following necessary conditions, as discussed in 3:

$$(1) \quad \frac{\partial u}{\partial c_0} = \frac{\partial u}{\partial c_1} \cdot [f'(k_1) + (1-\delta)]$$

$$(2) \quad f(k_0) + (1-\delta)k_0 = k_1 + c_0$$

$$(3) \quad f(k_1) + (1-\delta)k_1 = k_2 + c_1$$

$$(4) \quad k_2 = 0$$

3.2 Finite horizon model

In a finite horizon model, we have multiple time periods, such that $t = 1, ..T$. It's an arbitrary horizon, where $T > 0$. The problem is similar to the two period, where we want to optimize over all these time periods, deriving the same necessary conditions. We start with the resource identity is:

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$$

with constraints:

$$f(k_t) \geq 0, \quad f'(k_t) > 0, \quad f''(k_t) \leq 0 \quad \text{for all } k_t \geq 0, \quad \text{and } f(k_t) = 0 \text{ for } k_t = 0.$$

The household objective function is to maximize utility:

$$U_0 = \sum_{t=0}^{T} \beta^t u(c_t)$$

where $u'(c_t) > 0$, $u''(c_t) < 0$, and $\beta > 0$. Next we look at labor supply, which is exogenous and assumed to be perfectly inelastic with $n_t = 1$.

We solve:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to:

$$(1) \quad k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, \quad (2) \quad k_{t+1} \geq 0, \quad (3) \quad c_t \geq 0 \quad \forall 0 \leq t \leq T$$

Here k_0 is given (predetermined, or endowed), so the state variable is: k_t, and the control variables are: c_t, k_{t+1}.

$$\mathcal{L}_{\{c_t, k_{t+1}\}_{t=0}^{T}} = \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t \left(f(k_t) + (1-\delta)k_t - c_t - k_{t+1}\right) - \mu_{t+1}k_{t+1}\right]$$

FOCs:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} = 0 &\implies \beta^t u'(c_t) = \lambda_t \\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 &\implies \lambda_{t+1}\left(f'(k_{t+1}) + (1-\delta)\right) - \lambda_t - \mu_{t+1} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 &\implies f(k_t) + (1-\delta)k_t - c_t - k_{t+1} = 0 \\
\frac{\partial \mathcal{L}}{\partial k_{T+1}} = 0 &\implies -\lambda_T - \mu_{T+1} = 0 \Rightarrow \beta^T u'(c_T) = \lambda_T
\end{aligned}
$$

From the final first order condition, we know that $\beta^T u'(c_T) = \lambda_T$, so this implies that $\lambda_T k_{T+1} = 0 \Rightarrow k_{T+1} = 0$. This is because λ_t is the shadow price of investment. And therefore, in the final period, $T+1$, the value of investment or capital must be exhausted. Now, we combine the remaining first order conditions and rearrange to give us the (1) euler equation and (2) the capital accumulation conditions. These are both still **necessary conditions**, as is in the case of the two period model. Taking the first partial derivative, $\beta^t u'(c_t) = \lambda_t$:

$$
\begin{aligned}
u'(c_t) &= \beta u'(c_{t+1})\left(1 - \delta + f'(k_{t+1})\right) \\
k_{t+1} &= f(k_t) + (1-\delta)k_t - c_t \\
k_{T+1} &= 0 \\
k_o &= \text{ given}
\end{aligned}
$$

Sufficient conditions for a maximum:

- Compactness of feasible allocations:
  - $k_{t+1} \geq 0$
  - $c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$
  - $k_0 \geq 0$ and is finite
  - $f(k)$ is continuous for all $k \geq 0$
- $U_0(c_0, \ldots, c_T)$ is continuous in its arguments
- To ensure the necessary conditions are sufficient for a maximum, we can also ensure that the feasible allocation set is concave where $f(k)$ is concave and also that the utility function is concave.

3.3 Infinite horizon model

So now for an infinite horizon model, we must consider a few more important additions to the model. The key one is the addition of the transversality condition, which ensures that *the set of possible choices is compact*. This is an essential component of the model that makes the existence of the solution possible. The second key requirement is that the objective function is continuous. For clarity we write them out:

- **Transversality condition**: Every sequence of control variables, c_t, k_{t+1}, that satisfies the constraints to the optimization problem will converge. The set of all sequences is a closed subset of the real-valued sequence. This implies the feasible set is compact. Mathematically, it is written as:

  $$\lim_{T\to\infty} \beta^T \cdot u'(c_T) k_{T+1} = 0$$

  In its economic interpretation, it implies that no capital is ever left unused which could have increased utility (and welfare) via consumption. It's both intuitive and essential for the boundary behavior for a solution to exist.

- **Continuity of the objective function**: Where the limit of the utility function, as $T \to \infty$ will equal zero. This means the limit converges (does not go to ∞), only where $\beta < 1$. It would not be possible to have a continuous function of the feasible choices if the objective function diverged. It's written mathematically as:

  $$\lim_{T\to\infty} U_0 = \sum_{t=0}^{T} \beta^t u(c_t) \to 0$$

In economic terms, it means utility, super far into the future, converges to zero, due to discounting. This means the function is continuous over a set of feasible choices.

Mechanism to finding a system of equations to the solution of the infinite horizon model in a one-output model:

1. Set up Lagrange subject to the given constraints in the model. We start by writing the *household's objective function*:

$$U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t) = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t u(c_t)$$

Next we then see the *optimization problem* becomes:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t. } (1) \quad k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \quad (2) \quad k_{t+1}, c_t \geq 0$$

We get the Lagrange set up[3] to be:

$$\mathcal{L}\left(\{c_t, k_{t+1}, \lambda_t, \mu_t\}_{t=0}^{\infty}\right) = \lim_{T\to\infty} \sum_{t=0}^{T} \left[\beta^t u(c_t) + \lambda_t \left(f(k_t) + (1-\delta)k_t - c_t - k_{t+1}\right) - \mu_{t+1}k_{t+1}\right]$$

2. Derive the necessary conditions for a solution, including:

   (a) the Euler equation:
   $$u'(c_t) = \beta \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + (1+\delta)]$$

   (b) the resource identity:
   $$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$$

   where:

   - k_t is the capital stock at time t,
   - k_{t+1} is the capital stock at time $t+1$,
   - $f(k_t)$ is the output produced from capital at time t,
   - c_t is the consumption at time t,
   - δ is the depreciation rate of capital.

   (c) the resource constraint: $k_{t+1} \geq 0$

   (d) the (binding) constraints on the Kuhn-Tucker conditions for the initial condition, where k_0 is given:

   $$k_0 \geq 0$$

   $$k_t \geq 0$$

   $$c_t \geq 0$$

3. We then can get the Kuhn-Tucker conditions[4], that are able to give us important properties of the constraints in their limits. We have:

$$(1) \lim_{T\to\infty} k_{T+1} \geq 0 \quad (2) \lim_{T\to\infty} -\lambda_T \leq 0 \; (3) \lim_{T\to\infty} \lambda_T k_{T+1} = 0$$

These K-T conditions give us the **transversality condition**, such that:

$$\lim_{T\to\infty} \beta^T u'(c_T) k_{t+1} = 0$$

[3] We can drop the term $\mu_{t+1}k_{t+1}$ but apply the Kuhn-Tucker conditions. This allows us to get the first order conditions.

[4] Recall what the Kuhn Tucker conditions are the necessary conditions in a constrained optimization problem when we have non-binding constraints, i.e. inequalities in constraints. The KKT conditions tell us that the complementary slack condition must hold, meaning that either the multiplier is non-zero (inequality) or multiplier is zero (binds). Additionally, the sign of the Lagrange multiplier must be in the *same direction (positive)* as we are maximizing. It cannot be slack in the opposite direction.

4. Identify the sufficient conditions for an infinite time horizon model, including:

$$\begin{aligned} u(c_t) &\geq 0 & c_t &\geq 0 \\ f(k_t) &\geq 0 & k_t &\geq 0 \\ u''(c_t) &\leq 0 & f''(k_t) &\leq 0 \end{aligned}$$

Recall the sufficiency conditions *guarantee the existence of a solution.* They do this through proving two things: (1) the **continuity of the objective function** and (2) the **compactness of the feasible choice set**. The *Inada conditions* provide shorthand for these two conditions that insure interior long-run solutions. They ensure that capital doesn't fall to zero or accumulate, by applying the principles that there are positive and diminishing marginal returns to capital (as seen in the limit). These conditions ensure that capital in the long run converges, so we can have a stable steady state.

5. Linearize the model to get a system of differential equations that can approximate a linear solution around the steady state (c^*, k^*). The steady state equations are such that $c_t = c_{t+1} = c_{t+2}$ and so on, for all values of c. And similarly for capital. We therefore have the steady state equations:

$$c^* = f(k^*) - \delta k^*$$
$$f'(k^*) = \delta + (\frac{1}{\beta} - 1)$$

Here we linearize with respect to a small change from the previous period to the steady state, to understand the behavior in this neighborhood. We start by defining this system of equations around the steady state and the given parameters:

$$\varDelta c_{t+1} = c_t - c^*$$
$$\varDelta k_{t+1} = k_t - k^*$$

6. After we are able linearly approximate the system, we can re-write the solution as a matrix. This solution determines how capital stock and consumption evolve over time for any initial conditions (c_0, k_0):

The general solution for the linearized system is given by

$$\begin{bmatrix} c_{t+1} - c^* \\ k_{t+1} - k^* \end{bmatrix} = \begin{bmatrix} 1 - \beta \dfrac{c^*}{\sigma(c^*)} f''(k^*) & \dfrac{c^*}{\sigma(c^*)} f''(k^*) \\ -1 & \dfrac{1}{1+\rho} \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix},$$

where we write it in terms of $c_{t+1} - c^*$ and $k_{t+1} - k^*$ on the left-hand side.

7. From the linear solution, we want to solve for the eigenvectors and then eigenvalues, to get the slopes of the eigenvectors. Slopes of eigenvectors are then used to determine stability of a system given an initial condition (k_0). Using the eigenvectors and eigenvalues, we can express any path as:

$$\begin{bmatrix} c_{t+1} - c^* \\ k_{t+1} - k^* \end{bmatrix} = [\alpha \nu_-(1 + \lambda_-) + (1 - \alpha)\nu_+(1 + \lambda_+)] \, (k_t - k^*)$$

where:

$$\alpha \nu_- + (1 - \alpha)\nu_+ = \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}.$$

From this, we see that α is the only value that needs to be checked for whether it satisfies the transversality condition. We check both cases, $\alpha = 0$ and $\alpha = 1$. We get that only for $\alpha = 0$ do we get an optimal, unique solution s.t.

$$c_t - c^* = (1 + \lambda_-)^t m_-(k_0 - k^*)$$
$$k_t - k^* = (1 + \lambda_-)^t (k_0 - k^*)$$

So we get that $c_0 - c^* = m_-(k_0 - k^*)$ solves for c_0 and the optimal growth path is a stable, saddle path with asymptotic convergence from the initial values of capital and consumption towards the steady state values.

3.3.1 Model extensions

Comparison with competitive equilibrium: While not a true extension, the command economy can be compared to the competitive equilibrium (market equilibrium) to illuminate key difference between the two optimization strategies. While the strategy remains the same - *maximize utility* - we have a new constraint. This yields slightly different necessary conditions. In a competitive equilibrium (in a two-period model), we assume the household has a budget, where W_0 is the initial endowment, or wealth of a household:

$$W_0 = c_0 + p_1 c_1$$

and a constraint on labor supply:

$$0 \leq n_t \leq 1$$

Where W_0 is the initial household wealth, and the agent must maximize its utility over consumption in the two periods, with respect to the prices, p_1 and initial wealth, W_0. Now the first order conditions yield the following results:

$$\frac{\frac{\partial u}{\partial c_0}}{\frac{\partial u}{\partial c_1}} = \frac{1}{p_1} \quad \text{and} \quad n_0, n_1 = 1$$

From this comparison we see that the complexity of the problem increases. Here there are firms, households and capital that is owned by households but rented to firms. The households also supply labor to firms. Firms, similar to households are modeled identically as representative agents. In a market equilibrium, the investment and savings tradeoff will be the same in all periods to the command economy (when no distortions, such as taxes or transfers, are present).

Deterministic model with unanticipated one time gifts These models take into a one-time shock on productivity, A, where

$$A_t = A + \varDelta A$$

and the maximization problem becomes:

$$L = \sum_{t=0}^{\infty} \left[\beta^t u(c_t, \ell_t) + \lambda_t \left(Af(k_t, n_t) + (1-\delta)k_t - c_t - k_{t+1}\right) + \eta_t^1 \ell_t + \eta_t^2 n_t + \gamma_t(1 - n_t - \ell_t) + \mu_t k_{t+1}\right]$$

given the following constraints:

$$\begin{aligned}
&k_{t+1} - k_t = Af(k_t, n_t) - \delta k_t - c_t \\
&n_t + l_t \leq 0 \\
&n_t \geq 0, l_t \geq 0, k_t \geq 0 \\
&k_0 = \text{ given}
\end{aligned}$$

the first order conditions become:

$$\begin{aligned}
\frac{\partial L}{\partial c_t} &= 0 = \beta^t u_{c_t}(c_t, l_t) - \lambda_t \\
\frac{\partial L}{\partial n_t} &= 0 = \beta^t u_{l_t}(c_t, l_t) + \lambda_t (A \cdot f_{n_t}(k_t, n_t)) + \eta^2 - \gamma_t \\
\frac{\partial L}{\partial k_t} &= 0 = \lambda_t (A \cdot f_{k_t}(k_t, n_t) + (1-\delta)) - \lambda_{t-1} + \mu_t
\end{aligned}$$

from these FOCs, we can get the necessary conditions:

1 Euler Equation:

$$\beta^t \cdot u'(c_t, l_t)(A \cdot f(k_t, n_t) + 1 - \delta) = \beta^{t-1} u'(c_t, l_t) \implies$$

$$\beta \left(\frac{\partial u}{\partial c_{t+1}} (Af(k_{t+1}, n_{t+1}) + 1 - \delta) \right) = \frac{\partial u}{\partial c_t}$$

2 Labor-leisure tradeoff:

$$\frac{\partial u}{\partial c_t} (A \cdot f_{n_t}(k_t, n_t)) = \frac{\partial u}{\partial l_t}$$

3 Resource identity (taken from the constraints, as is):

$$k_{t+1} = Af(k_t, n_t) + (1-\delta)k_t - c_t$$

4 Transversality condition:

$$\lim_{t\to\infty} \beta^t \left[\frac{\partial u}{\partial c_t} \cdot k_{t+1}\right] = 0$$

Now we take a look at the stead state dynamics, and the effect of a shock on productivity (A) on the rest of the economic variables. The steady state, (k^*, c^*, n^*) is given by

$$A\frac{\partial f(k^*, n^*)}{\partial k} - \delta = \rho,$$

$$Af(k^*, n^*) - \delta k^* = c^*,$$

$$c^* = (1-n^*)^\gamma A\frac{\partial f(k^*, n^*)}{\partial n}$$

Then we differentiate

$$A\frac{\partial^2 f(k^*, n^*)}{\partial k^2}dk + A\frac{\partial^2 f(k^*, n^*)}{\partial k\partial n}dn + \frac{\partial f(k^*, n^*)}{\partial k}dA = 0,$$

$$A\frac{\partial f(k^*, n^*)}{\partial k}dk - \delta dk + A\frac{\partial f(k^*, n^*)}{\partial n}dn + f(k^*, n^*)dA = dc$$

$$dc = (1-n^*)^\gamma\left(\frac{\partial f(k^*, n^*)}{\partial n}dA + A\frac{\partial^2 f(k^*, n^*)}{\partial n^2}dn + A\frac{\partial^2 f(k^*, n^*)}{\partial k\partial n}dk\right) - \gamma(1-n^*)^{\gamma-1}A\frac{\partial f(k^*, n^*)}{\partial n}dn$$

Now we can substitute in the FOCs to simplify:

$$A\frac{\partial^2 f(k^*, n^*)}{\partial k^2}dk + A\frac{\partial^2 f(k^*, n^*)}{\partial k\partial n}dn = -\frac{\rho+\delta}{A}dA,$$

$$\rho dk + \left(\frac{c^*}{(1-n^*)^\gamma}\right)dn - dc = -f(k^*, n^*)dA,$$

$$(1-n^*)^\gamma\left(A\frac{\partial^2 f(k^*, n^*)}{\partial n^2}dn + A\frac{\partial^2 f(k^*, n^*)}{\partial k\partial n}dk\right) - \gamma\frac{c^*}{1-n^*}dn - dc = -\frac{c^*}{A}dA.$$

Now, solving the three derivatives (each of the state steady variables, (c^*, k^*, n^*), with respect to A), we can understand how the steady state values of our variables of interest will change with respect to the productivity shock.

$$\begin{bmatrix}\frac{dk}{dA}\\ \frac{dn}{dA}\\ \frac{dc}{dA}\end{bmatrix} = \begin{bmatrix} A\frac{\partial^2 f(k^*, n^*)}{\partial k^2} & A\frac{\partial^2 f(k^*, n^*)}{\partial k\partial n} & 0 \\ (1-n^*)^\gamma A\frac{\partial^2 f(k^*, n^*)}{\partial k\partial n} & (1-n^*)^\gamma A\frac{\partial^2 f(k^*, n^*)}{\partial n^2} - \gamma\frac{c^*}{1-n^*} & -1 \\ \rho & \frac{c^*}{(1-n^*)^\gamma} & -1\end{bmatrix}^{-1}\begin{bmatrix}-\frac{\rho+\delta}{A}\\ -f(k^*, n^*)\\ -\frac{c^*}{A}\end{bmatrix}.$$

To interpret this, we can think about shocks to productivity as:

$$\begin{aligned} &\text{in } t=0 \to A_0 = A \\ &\qquad\quad \to A_1 = A + dA \text{ temporary, one period shock} \\ &\text{in } t \geq 1 \to A_t = A \text{ return to normal level of } A \end{aligned}$$

And with consumption smoothing, we act as though this shock is a one time, unanticipated gift. As such, The period after the shock ($t > 1$), we have more capital because we are on a consumption path that has higher capital in $t+1$ (given the law of capital motion: $k_{t+1} - k_t = (A + dA)f(k_t, n_t) + \delta k_t - c_t$. So, when there is a *positive shock*, we choose to consume some of the additional gains and invest some of them in order to smooth consumption over time (perfect foresight).

Learning and habit formation

Here we can see how people, when they get a shock, will respond over multiple shocks. See **Q2 MACRO NOTES** for a more detailed discussion.

Time variant β

Where rate of time preference can change in future periods. This was not covered in the course, but its something to consider. See hyperbolic discounting in **Q3 MICRO NOTES.**

Introduction of labour markets

Where utility now depends on both consumption and leisure, while labour is an input into the production function. We assume we're still in a single-good model (i.e. we produce only one output).

Maximize:

$$\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

subject to:

$$(1) \quad Y_t = f(k_t, n_t) \qquad (2) \quad k_{t+1} = Y_t + (1-\delta)k_t + i_t$$

with Lagrange:

$$L = \beta^t u(c_t, l_t) - \lambda_t(f(k_t, n_t) - (1-\delta)k_t - c_t + k_{t+1})$$

Now this model can end here. Or, consequently, it can also be extended to include the accumulation of wealth, as seen in the next section, 3.4.

Single sector with adjustment costs

Here we use a standard infinite horizon model with one sector of output. We assume there exists some **cost** to making an investment in the economy, such that we have to modify the resource identity (law of capital motion) to capture this adjustment cost. The new set up would be:

$$\max U(x) \text{ s.t. } k_{t+1} - k_t = i_t - c_t + \varphi(i_t, k_t)$$

Here, we normally assume there are quadratic adjustment costs, such that $i_t \left(\frac{\varnothing}{2}, \frac{i_t}{k_t} \right)$, where k_t, i_t are proportional in the state state and there exists some cost to holding onto the capital into the next period. This makes the constraints for a command optimum, with adjustment cost:

$$\begin{aligned}
&k_{t+1} - k_t = i_t - \delta k_t \\
&f(k_t, n_t) = i_t + c_t + \varphi(i_t, k_t) = i_t(1 + \frac{\varnothing}{2} + \frac{i_t}{k_t}) + c_t \\
&k_{t+1} \geq 0 \\
&k_0 = \text{ given}
\end{aligned}$$

Putting this all together, we do the same approach of setting up the Lagrange, and then deriving the FOCs. From the FOCs we can find the Euler equation. When we do so, we solve for λ_t which has a unique interpretation here, such that it is no longer the shadow price of capital, set equivalent to the marginal utility of consumption. The interpretation here is different, such that λ_t is the shadow price of an additional unit of investment:

$$\mathcal{L} = \max \beta^t \left[U(c_t, 1-n_t) - \lambda_t(i_t - \delta k_t) - \mu_t(i_t(1 + \varphi(i_t, k_t)) + c_t) \right]$$

FOCs:

$$\begin{aligned}
&\implies \mu_t = u'_{c_t}(c_t, 1-n_t) \\
&\implies \lambda_t = \mu_t(1 + \frac{\varnothing}{2} + \frac{i_t}{k_t}) + \frac{1}{k_t} \to \frac{\lambda_t}{u'_{c_t}(c_t, 1-n_t)} \\
&\qquad = 1 + \frac{\varnothing}{2} + i_t
\end{aligned}$$

3.4 Yeomani peasant

This model is a autarkic or planner-based model, where there are multiple agents, and they can store (accrue debt) assets against their future labor income. This is an important departure from a standard command economy model as we impose a solvency condition on agents. This model's framework is what we use for the permanent income hypothesis. We will get more into the details here.

3.4.1 Find constraints

- Simple budget identity[5]:

$$a_{t+1} = (1 + r_t)a_t + w_t n_t - c_t$$

We need to make an **intertemporal budget constraint** by using the state variables, a_o and the state budget identity (above). From this, and through iteration, we can get a budget identity that works for all periods. The steps are as follows:

$$\begin{aligned}
(a_{t+1})(1 + r_t)^{-1} &= a_t + w_t n_t (1 + r_t)^{-1} - c_t (1 + r_t)^{-1} \\
a_{t-1} &= a_t + w_{t-1} n_{t-1} - c_{t-1}
\end{aligned}$$

$$\begin{aligned}
&\text{such that } a_0 = a_t \text{ is given, so we can substitute in for } a_t \\
(a_{t+1})(1 + r_t)^{-1} &= a_t + [w_{t-1} n_{t-1} + w_t n_t (1 + r_t)^{-1}] - c_{t-1} + c_t (1 + r_t)^{-1} \\
&= \sum_{t=0}^{t} w_t n_t (1 + r_t)^{-t} - \sum_{t=0}^{t} c_t (1 + r_t)^{-t} \\
&= \sum_{s=0}^{t} w_s n_s (\frac{1}{1+r})^s - \sum_{s=0}^{t} c_s (\frac{1}{1+r})^s + (1+r)a_0 \\
a_{t+1}(\frac{1}{1+r})^t &= \sum_{s=0}^{t} w_s n_s (\frac{1}{1+r})^s - \sum_{s=0}^{t} c_s (\frac{1}{1+r})^s + (1+r)a_0
\end{aligned}$$

- **Solvency condition:** The solvency condition, sometimes called the No Ponzi condition on equilibrium means an agent must pay off their debts and are restricted from carry debts beyond their lifetime. Therefore, during their lifetime, negative wealth is possible. But, when agents have monotonistic preferences, there is a limit to how much borrowing can occur if just to increase consumption.

$$\lim_{t \to \infty} (\frac{1}{1+r})^t \cdot a_{t+1} \geq 0$$

This constraint implies the following in equality:

$$= (1 + t)a_0 + \sum_{t=0}^{t} (\frac{1}{1+r})^t w_t n_t - \sum_{t=0}^{t} (\frac{1}{1+r})^t c_t$$

Where the first term, $(1 + r)a_0$ is initial wealth by an agent. The second term is the present value flow of labor income, and the third term is the present value flow of consumption (where prices are constant).

- **Resource constraint:** Households are constrained on how much consumption they may undertake in each period by their wealth endowment, a_0. This constraint shows that wealth can grow at a rate up to r, but no greater. The constraint is also a summation, such that:

$$\sum_{t=0}^{\infty} (\frac{1}{1+r})^t \cdot c_t \leq (1 + r)a_0 + \sum_{t=0}^{\infty} (\frac{1}{1+r})^t \cdot w_t n_t \leq 0$$

3.4.2 Model set up

Now we maximize a households' utility over the resource constraint, using a Lagrange approach to derive the first order conditions and necessary conditions:

$$\max U(c_t, (1 - n_t)) = \sum_{t=0}^{\infty} \beta^t u(c_t, (1 - n_t))$$

[5] The budget identity, also the equation of motion here, is written in the *state* form. We could similarly rewrite it with respect to flow variables as:

$$a_{t+1} - a_t = (r_t)a_t + w_t n_t - c_t$$

. Here we are looking at the flow of savings (difference in savings) from one period to the next. This budget identity is not the law of capital motion, but the law of motion for the assets, as there is no capital in this model.

s.t.

$$a_0 > 0$$
$$a_{t+1} = (1+r)a_t + w_t n_t - c_t$$
$$\lim_{t\to\infty} (\frac{1}{1+r})^t a_{t+1} \geq 0$$
$$n_t \geq 0, (1-n_t) \geq 0$$

with the Lagrange:

$$\mathcal{L} = \beta^t(u(c_t, 1-n_t)) - \lambda_t[a_{t+1} + (1+r)a_t - w_t n_t + c_t]$$

FOCs:

$$\frac{dL}{dc_t} = 0 = \beta^t \frac{dU}{dc_t} - \lambda_t$$
$$\frac{dL}{dn_t} = 0 = \beta^t \frac{dU}{d1-n_t} - \lambda_t w_t$$
$$\frac{dL}{da_{t+1}} = 0 = -\lambda_t + \lambda_{t+1}(1+r)$$
$$\frac{dL}{d\lambda_t} = 0 = a_{t+1} + (1+r)a_t - w_t n_t + c_t$$

Rearranging these FOCs we get the following:

$$\lambda_t = \beta^t \frac{dU}{dc_t} \implies \lambda_{t+1} = \beta^{t+t} \frac{dU}{dc_{t+1}}$$

Now plug into FOC from $\frac{dL}{da_{t+1}}$, such that we get the **euler equation**:

$$\frac{dU}{dc_t} = \beta \frac{dU}{dc_{t+1}}(1+r)^t$$

Now get the **labor-leisure tradeoff** from from the FOC for $\frac{dL}{dc_t}$ and $\frac{dL}{d1-n_t}$ such that:

$$\frac{du}{dn_t} = wt \cdot \frac{du}{dc_t}$$

3.4.3 Solve from c_0

- Take euler equation, and set the variables at $t = 0$, so the equation is: $\frac{dU}{dc_t}(c_0, 1-n_0) = \beta \frac{dU}{dc_{t+1}}(c_t, 1-n_t)(1+r)^t$
- Rearrange to isolate $(\frac{1}{1+r})^t$
- Plug into solvency condition:

$$\lim(\frac{1}{1+r})^t a_{T+1} \geq 0$$
$$\lim(\beta^t)(\frac{du}{dc_t})(a_{t+1}) = 0$$
$$\lim(\beta^t)(\beta^t(1+r)^t \frac{du}{dc_t})(a_{t+1}) = 0$$
$$\lim(\frac{1}{1+r})^t(\frac{du}{dc_t}(c_0, 1-n_0))(a_{t+1}) = 0$$

This shows that solvency holds, and the sum of financial wealth will equal to zero where there is no remaining capital in the economy. This no ponzi condition implies rationality of agents (*similar to Nash equilibrium behavior*).

From this we see that households have perfect foresight, and prices are relative in the future period. By iterating back from c_0 we can get the optimal path for consumption:

$$c_t^* = (1-\beta)\left[(1+r)a_t + \sum_{s=1}^{t}(\frac{1}{1+r})^t w_t\right]$$

There is a possibility of extending the model, such that we can look at ρ[6], and the effect of a different discount rates on the optimal consumption path. We can look at three cases:

$$\beta(1+r) > \rho \tag{1}$$

$$\beta(1+r) < \rho \tag{2}$$

$$\beta(1+r) = \rho \tag{3}$$

In the first case, c_t grows over time. In the second case, c_t declines as today is favorable for deriving value from consumption. In the case that $\rho = r$, then $\beta = \dfrac{1}{1+\rho} = \dfrac{1}{1+r}$. In this third case, total wealth, $\overline{w}$ is found from the equation for c_0 such that $c_0 = (1-\beta)[(1+r)a_0 + \overline{w}]$. Here, consumption is constant, and is a function of total wealth. We refer to this as **permanent labor income.**

3.5 Assessing dynamics

In order to assess the dynamics of these models, we often need to consider how initial endowments in these economy affect the equilibrium path for optimal growth. These optimal growth paths, are considered as a linearized functions around steady state points.

Why do we linearize? The linearization (see 6.1 for step by step) gives us a good approximation of the dynamics of the non-linear model in the neighborhood of the steady state. We use it as this method gives us the same dynamic properties as the original non-linear system, in the neighborhood of the steady state, but makes it simpler for us to interpret the changes. We are able to do this by getting the following:

- An **equation** for the difference between the current state along the optimal pathway and the steady state, expressed as[7]:

$$dc_t = c_t - \overline{c}$$

- The **slope** of the pathway of the variables of interest, such as c_t. We find the slope by taking the partial derivative of the first order necessary conditions *(euler equation, law of motion of capital)* with respect to our variable of interest. This gives us second derivatives of the initial equation that tell us the slope of the line at our variables of interest that we are trying to linearize.

- The method uses a **Taylor expansion** when working with a system of equations that defines changes in variables from one period of time to another (the marginal utility in two different periods, or the marginal capital). Taking this first-order expansion gives us linearization around the steady state, and approximates a dynamic system. In general form:

$$\triangle y_{t+1} = a_{11}y_t + a_{12}x_t$$
$$\triangle x_{t+1} = a_{21}y_t + a_{22}x_t$$

  where:

$$y_t = (c_t - c^*)$$

$$x_t = (k_t - k^*)$$

[6] Recall $\rho = \dfrac{1-\beta}{\beta}$, such that $\beta = \dfrac{1}{1+\rho}$.

[7] Note that it can also be written in the following ways, depending on preference. It is important to note that the d, when used, does indicate the *difference* between the two point, while the triangle represents the change between the two. We can use them indiscriminately:

$$\triangle c_t = c_t - \overline{c}$$
$$dc_t = c_t - c^*$$
$$\triangle c_t = c_t - c^*$$

From this, we get a system of linear equations that are differential equations, where we create a colinear space. The system is:

$$\begin{bmatrix} \triangle y_{t+1} \\ \triangle x_{t+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} y_t \\ x_t \end{bmatrix}$$

We then label these matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$V = \begin{bmatrix} x \\ y \end{bmatrix}$$

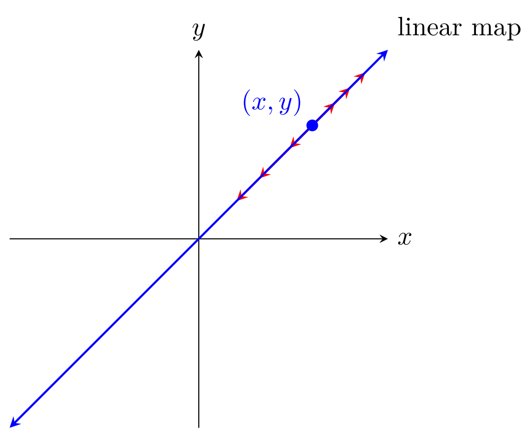
Putting it together to solve for the eigenvalues:

$$\begin{gathered} (A - \lambda I)V = 0, \quad det(A - \lambda I) = 0 \implies \\ (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} = 0 \\ DetA - \lambda(a_{11} + a_{22}) + \lambda^2 = 0 \end{gathered}$$

Where the latter two equations are the roots of the characteristic polynomial, such that we can get the eigenvalues (where V are the eigenvectors) by solving for the λ *(use quadratic formula)*:

$$\begin{aligned} \lambda_-, \lambda_+ &\implies \\ \lambda V &= AV \\ \lambda_+ V_+ &= AV_+ \\ \lambda_- V_- &= AV_- \end{aligned}$$

- Where the **phase diagram** can show us that a vector of the linear map from $(x_t, y_t) \to (x_{t+1}, y_{t+1})$. We can think of these as the preliminary introduction to the eigenvectors which serve as the basis for transformation of this linear system of equations.



- After we identify the linear system, and derive a set of eigenvectors, we can then look at a complete phase diagram such that we can determine which way a point is trending. The sign of these eigenvectors approximates the solution around the optimum and tells us if an equilibrium value, the vector c^*, k^*, will converge to the steady state.

Through linearization, we learn about the behavior of the equilibrium choice variables in the following cases:

- Responses to shocks (productivity, income), along the equilibrium path
- Behavior around the steady state of key choice variables

4 Dynamic stochastic general equilibrium models

Dynamic General Equilibrium (DSGE) models incorporate the temporal dimension, analyzing how economic variables evolve over time. They combine the comprehensive market interactions of GE models with dynamic optimization by agents. We will solve these models sequentially, i.e. over all periods, t.

One type of DSGE model is the **Arrow-Debreu style general equilibrium models**, such that there are many agents, and they are able to trade. These model represent general equilibrium in which relative prices exist and markets must clear simultaneously, and are subsequently **not autarky**. This is because trade is allowed. Most importantly, pricing exists. Goods have prices. Households have income, and can also have debt, for which they can borrow against (subject to conditions).

4.1 Integrating firms

Now this isn't its own model in itself. But, we were taught about the theory of the firms in macro models, and this is important for integrating firms into the fiscal policy shock model. We will need know the *firm's optimization problem*. We can show that a firm, in one sector, optimizes with respect to its investments. We assume all firms are homogenous and they receive some capital and investment. This allows us to get a value for capital.

We start with firms that are producing and consuming with different types of goods. Firms choose to maximize their value function, V_0

$$\max_{\{n_t,i_t,k_{t+1},b_{t+1},x_t\}_{t=0}^{\infty}} V_0 = \frac{1}{1+r_0}\sum_{t=0}^{\infty} R_t^{-1}(f(k_t,n_t) - w_t n_t - i_t + x_t)$$

Subject to the necessary conditions:

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + i_t \\ b_{t+1} &= (1+r_0)b_t + i_t \\ k_0, b_0 &\rightarrow \text{ given} \\ k_{t+1} &\geq 0 \\ \lim_{t\to\infty} R_t^{-1} b_{t+1} &= 0 \end{aligned}$$

The first step is to take the summation of the budget constraint on bonds, b_t in order to understand why we end up with the necessary condition above. We start by solving:

$$\sum_{t=0}^{\infty} R_t^{-1}(b_{t+1} - (1+r_t)b_t) \implies (1+r_0)b_0$$

Adding in adjustment costs: If we want to add in adjustment costs, then we are maximizing the value function V_0 and the bond holdings in $t=0$ such that we have:

$$\max_{\{n_t,i_t,k_{t+1},b_{t+1}\}_{t=0}^{\infty}} V_0 + b_0 = \frac{1}{1+r_0}\sum_{t=0}^{\infty} R_t^{-1}\left(f(k_t,n_t) - w_t n_t - (1 + \frac{\Phi}{2}\frac{i_t}{k_t})i_t\right)$$

Now we have the adjustment cost in blue. Payments to V_0 come in the form of a dividend payment. As Such, we can finally rewrite such that:

$$\max_{\{n_t,i_t,k_{t+1}\}_{t=0}^{\infty}} (1+r_0)V_0 = \frac{1}{1+r_0}\sum_{t=0}^{\infty} R_t^{-1}\left(f(k_t,n_t) - w_t n_t - (1 + \frac{\Phi}{2}\frac{i_t}{k_t})i_t\right)$$

with the new variables and first order conditions:

$$\begin{aligned} R_t^{-1} q_t &\rightarrow \text{ market discount factor} \\ k_{t+1} = (1-\delta)k_t + i_t &\rightarrow \text{ resource constraint} \\ k_0 &\rightarrow \text{ given} \\ k_{t+1} &\geq 0 \\ w_t = f_n'(k_t,n_t) &\rightarrow \text{ wage price} \end{aligned}$$

$$f_k'(k_t,n_t) + (\frac{\Phi}{2})(\frac{i_t}{k_t})^2 + (1-\delta)q_{t+1} - (1+r_t)q_t = 0$$

We can find the **shadow price of capital**: $(1+\Phi\frac{i_t}{k_t}) = q_t$.[8] Rearranging again to isolate for i_t we can then substitute in $\frac{q_{t-1}}{\Phi}\cdot k_t$ in both the resource constraint on capital and the final first order condition. This gives us the following two equations:

$$k_{t+1} = (1-\delta)k_t + \frac{q_{t-1}}{\Phi}\cdot k_t$$

$$f'_k(k_t,n_t) + (\frac{\Phi}{2})(\frac{q_{t-1}}{\Phi})^2 + (1-\delta)q_{t+1} - (1+r_t)q_t = 0$$

Again, rearranging the final condition:

$$(1-\delta)q_{t+1} = (1+r_t)q_t - \frac{\Phi}{2}\left(\frac{q_{t-1}}{\Phi}\right)^2 + f'_k(k_t,n_t)$$

You then linearize for q_{t+1} and k_{t+1}.

$$\hat{q}_{t+1} = \frac{(1+\bar{r})}{1-\delta}\hat{q}_t + \frac{1}{1-\delta}\hat{r}_t - \frac{1}{(1-\delta)\bar{q}}\cdot\bar{q}\cdot\frac{\bar{q}}{\Phi}\hat{q}_{t-1} + \frac{\bar{k}f_{kk}(\bar{k},\bar{n})}{(1-\delta)\bar{q}}\hat{k}_t + \frac{\bar{n}f_{kn}(\bar{k},\bar{n})}{(1-\delta)\bar{q}}\hat{n}_t$$
$$\hat{k}_{t+1} = \frac{(1-\delta)}{1-\delta+\frac{\bar{q}}{\Phi}}\hat{k}_t + \frac{\frac{\bar{q}}{\Phi}}{1-\delta+\frac{\bar{q}}{\Phi}}\hat{q}_{t-1}$$

Dynamics: What we find is the following:

- q_t is a forward looking variable for the firm. Firm has an employment and investment plan. It's important to note that q_0 is not a control variable, but its an outcome of the control variables in the function,
- $\frac{\partial V_0}{\partial k_0} = q_0$, holding for any time period, t,
- When $k_{t+1} = k_t = k^*$, so in equilibrium, then $(1-\delta) + \frac{q_{t-1}}{\Phi} = 1 \implies \triangle k_{t+1} = 0$,
- Firms respond to supply shocks, $A\cdot f'_k(k_t,n_t)$ by increasing investment temporarily. This is the case without adjustment costs. Then, after the shock, investment settles back down. The longer the shock lasts, we will need to look at q_t to see how the firms will end responding

Recall in this problem we are able to deduce **Tobin's Q** as:

$$q_{-1} = \frac{V_0}{k_0}$$

Tobin's Q is defined as the book value of capital, less its depreciation. The next steps in this problem would be eliminate the time subscripts, and use this functional form with Bellman's equation to find the optimal value function. Given that this is covered in **Q2 NOTES** I will not go over the steps here.

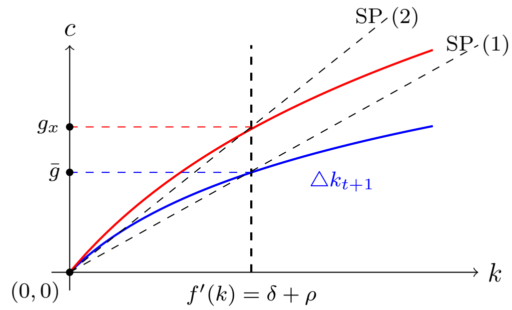
4.2 Ramsey growth with fiscal policy model

4.2.1 Model overview

In a Ramsey growth model with taxation, we are solving for a decentralized competitive equilibrium. We want to find a competitive equilibrium, that is also pareto efficient (where preferences are monotonic). With fiscal policy, the government imposes distortionary (labor, capital or consumption) taxes, or non-distortionary transfers.[9] The imposition of a distortionary tax results in *deadweight loss*, which is a wedge between firm's price of labor (wage) and what the household then receives for its labor. In this model, the government does not have their own objective function, instead they are maximizing household welfare, looking for the optimal growth path of consumption. We can assume, to start, that g_t is exogenous. Below we can see a quick visual of the dynamics, that shows government expenditure *does not* effect the optimal choice of consumption.

[8] Source on where this shadow price of capital comes from is still not clear to me. Going to update when I can find it.

[9] Note that transfers are not distortionary because it does not enter into the first order conditions. As such, it doesn't change the behavior of agents with respect to trade-offs such as labor and leisure.



Optimal government expenditure effect on households

Budgets: A Ramsey growth model can be solved for both the social planner's optimum and with the competitive equilibrium approach. There are now three budget identities, related to bonds, b_t, capital, k_t, and assets, a_t. These three budget identities are always redundant, as $a_t = k_t + b_t$.

Intuition: The intuition in this model is that households are price takers, for both market prices and wages. They also accept taxes as given, and optimize over their utility function. Firms rent capital from households, they optimize for profit, and take both prices and taxes as given. Then, government optimize with respect to household consumption optimization. All markets (goods, factor, and asset markets) must clear. A key distinction to remember is that we are not working in the command optimum framework, as we are not considering $v(g_t)$ enters the utility function. If $v(g_t)$ were in the utility function, we would consider this a *true public good*, where there is no rival and households gain utility from consuming it. In this case, there would be no government debt, no resource identity, and no equilibrium prices.

To solve here, there are two approaches. The first is a competitive equilibrium, where we find an allocation in a competitive equilibrium, stating, again, that g_t is exogenous. The competitive equilibrium, then must be restricted to the set of equilibria within the government's budget constraint. Let's define key terms for solving the model:

(1) Competitive equilibrium with distorting taxes: A competitive, decentralized or market equilibrium is an allocation and prices such that:

- Households maximize their utility against their budget, taking $\{w_t, r_t, \tau_t^n, \tau_t^k\}_{t=0}^{\infty}$ as given,
- Firm maximize their profits taking prices as given,
- The government's budget identity holds, and the constraint, expenditures must be less than or equal to revenues, also holds,
- All markets clear.

Secondly, we can use the Ramsey optimal taxation problem, where we solve, *from the social planner's perspective* for the optimal fiscal policy. This approach requires the **implementability condition**, such that the social planner integrates the household's first order conditions into the optimal fiscal policy sequence so that, under and equilibrium with taxation as given, the household would still find this allocation to be optimal. This is often called the *primal approach*, where the implementability condition is constructed from:

- the household's budget identity
- initial wealth allocation, a_0
- first order condition for labor-leisure choice
- euler condition

(2) Ramsey equilibrium: An allocation, prices, and fiscal policy (taxes) are given such that the allocation and prices satisfy a competitive equilibrium. Both households and firms take the fiscal policy as given, and the government maximizes household utility with respect to choice variables, $\{g_t, \tau_t^k, \tau_t^n, b_{t+1}\}_{t=0}^{\infty}$.
Under the first theorem of welfare economics, we know that a competitive equilibrium is a pareto optimum when preferences are monotone and the competitive equilibrium exists. The relationship between these two concepts does not hold in both directions, i.e. pareto optimum $\Longleftarrow \Longrightarrow$ competitive equilibrium is not true.

There are four types of taxes:

- Tax on consumption (τ_t^c)
- Tax on labor (τ_t^n)
- Tax on capital (τ_t^k)
- Transfer (negative) (T_t)

We will start with the unique, non-distortionary case of lump sum taxation. Then we will start by modeling a tax on labor and consumption, then separately modeling a tax on capital. These taxes can be combined. First I define key variables:

- a_t: total assets ($= k_t + b_t$)
- $R_t^{-1} = \Pi_{s=1}^t(\dfrac{1}{1+r_s})$: market discount factor, used in place of the stochastic discount factor as households are not discounting against their intertemporal tradeoff, but are accounting for returns to assets
- b_t: government bonds, which are *negative* in the government's budget constraint, because the government is a net borrower when they issue bonds to households
- w_t: wage rate per unit of labor
- r_t: time-dependent rate of return on assets
- δ: rate of depreciation of capital
- g_t: government spending

4.2.2 Lump sum taxation

To start, we have a representative household with the following utility function:

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(c_s)$$

There is an exogenously given stream of government expenditures, $\{g_t\}_{t=0}^{\infty}$. Now we want to **define a competitive equilibrium** - optimizing households and firms. In words: a competitive equilibrium is an allocation, $\{c_t, g_t, k_{t+1}\}_{t=0}^{\infty}$ and prices $\{w_t, r_t\}_{t=0}^{\infty}$ such that households max their utility subject to:

(1) budget constraint: $a_{t+1} = (1+r_t)a_t + w_t - T_t - c_t$

(2) solvency constraint: $\lim_{t\to\infty} R_t^{-1} a_{t+1} \geq 0$

(3) given initial assets, a_0

(4) and the market discount factor, $R_t^{-1} = \Pi_{s=1}^t(\dfrac{1}{1+r_s})$ where $R_0^{-1} = 1$

Next we have firms that maximize their profits. Firms production function is given as $y = f(k_t)$, and they optimize:

$$\max_{n_t^d, k_t^d} \left(n_t^d \cdot f\left(\frac{k_t^d}{n_t^d}\right) + w_t n_t^d - \nu_t k_t^d \right)$$

The third component is that markets (assets, goods, and factor markets) must clear in a competitive equilibrium:

(1) law of capital motion: $k_{t+1} = f(k_t) + (1-\delta)k_t - c_t - g_t$

(2) labor and capital markets (factor) clear: $n_t^s = n_t^d = 1$ and $k_t = k_t^d$

(3) asset markets clear: $a_t = b_t + k_t$

Finally, the government budget constraint must be satisfied using the budget identity, and the solvency constraint[10] for the government must hold:

$$b_{t+1} = (1+r_t)b_t + g_t - T_t \qquad \lim_{t\to\infty} R_t^{-1} b_{t+1} \leq 0$$

Now we **solve for the competitive equilibrium**. Start by taking the FOCs to find (a) the solvency condition, (b) factor market equilibrium conditions, (c) resource identity, and (d) the transversality condition. We will then have all necessary conditions for the equilibrium.

$$\mathcal{L} = \max \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) - \lambda(a_{t+1} - (1+r_t)a_t - w_t + c_t + T_t)$$

FOCs for the household:

$$\frac{\partial L}{\partial c_t} = 0 = \beta^{s-t} u'(c_s) - \lambda_t$$
$$\frac{\partial L}{\partial a_{t+1}} = 0 = -\beta^{s-t+1} \lambda_{t+1}(1+r_{t+1}) + \beta^{s-t}\lambda_t$$
$$\frac{\partial L}{\partial T_t} = 0 = -\beta^{s-t}\lambda_t$$

Rearranging we can get the euler equation and the asset identity:

$$(1) \quad \beta u'(c_{t+1})(1+r_{t+1}) = u'(c_t) \qquad (2) \quad a_{t+1} - (1+r_t)a_t - w_t + c_t + T_t$$

We also get that the transversality condition(s) holds in equilibrium:

$$\mathop{lim}_{t\to\infty} \beta^t u'(c_t) a_{t+1} = 0$$

Then solve the firm's optimization problem: max π with FOCs:

$$\frac{\partial \pi}{\partial n_t} = 0 = f\left(\frac{k_t^d}{n_t^d}\right) - f'\left(\frac{k_t^d}{n_t^d}\right) k_t^d + w_t$$
$$\frac{\partial \pi}{\partial k_t} = 0 = f'\left(\frac{k_t^d}{n_t^d}\right)\left(\frac{1}{n_t^d}\right) - \nu_t$$

We get the FOCs:

$$(1) \quad w_t = f(\cdot) - f'(\cdot)k_t \qquad (2) \quad \nu_t = f'(\cdot)\frac{1}{n_t} \implies \nu_t = (r_t + \delta)$$

This second condition for the rental price of capital holds because we can use the asset pricing identity, a_t, to see how capital and bonds are restricted over time. When we subtract b_t from a_t we are left with the asset identity with only the value of k_t in the equation. As such, we end up getting:

$$a_{t+1} = (1+r_t)a_t + w_t n_t - T_t - c_t$$

$$b_{t+1} = (1+r_t)b_t - T_t + g_t$$

$$\text{given } a_t = b_t + k_t \to a_{t+1} - b_{t+1} = k_{t+1}$$

$$k_{t+1} = (1+r_t)k_t + w_t n_t - c_t + g_t \equiv f(k_t, n_t) + (1-\delta)k_t - c_t + g_t$$

So when you take this result, combined with (2) derived from the firm's optimization problem, we take the equivalence $(1+r_t)k_t + w_t n_t = f(k_t, n_t) + (1-\delta)k_t$ and rearrange:

$$(1+r_t)k_t - (1-\delta)k_t + w_t n_t = f(k_t, n_t)$$

$$(r_t + \delta)k_t = f(k_t, n_t) - w_t n_t$$

[10] See that the solvency constraint is that the government's debt is a *negative term*. As such, the government should hold a positive amount of assets, so we want the limit to be less than zero. This ensures that negative debt is reduced to zero in the long run. Written in a different way, we could see the solvency as $lim\ R_t^{-1} a_{t+1} \geq 0$.

now take first derivative wrt k_t[11]

$$\frac{\partial}{\partial k_t} = -(r_t + \delta) + f'(\cdot) \implies (r_t + \delta) = f'(\cdot)$$

This result is also directly due to the government budget identity, $b_{t+1} = (1 + r_t)b_t - T_t + g_t$, household budget identity $a_{t+1} = (1 + r_t)a_t + w_t - T_t - c_t$, and asset market clearing condition $a_t = b_t + k_t$.[12] This also tells us, upon closer look, that the rate of return on the government's debt, r_t is equivalent to the return on the marginal productivity of capital, less the depreciation, such that

$$\begin{aligned}
r_t &= f'_k(\cdot) - \delta \\
r_t &= \nu_t - \delta \\
r_t &= (1 - \tau^k_t)\nu_t - \delta \text{ where capital income is taxed}
\end{aligned}$$

In the scenario in which capital is taxed in a decentralized economy, we take the after tax rate of return on capital, ν_t, less depreciation.

From the findings above, we then use capital, k_{t+1} in the identity for the law of capital motion. Below, we are simply solving from the government's perspective to find an equilibrium condition that clears the market, is constrained by the government's budget identity and is an optimal value for firms and households. So, we get k_t:

$$k_{t+1} = (1 + r_t)k_t + w_t - g_t - c_t \implies k_{t+1} = f(k_t) - (1 - \delta)k_t - g_t - c_t$$

Finally we get the optimal pathway for bonds, $b_{t+1} = (1 + r_t)b_t + g_t - T_t$. And we get the transversality condition,

$$\lim_{t\to\infty} \beta^t u'(c_t)(k_{t+1} + b_{t+1}) = 0$$

From the above, in a competitive equilibrium $\{c_t, k_{t+1}\}^{\infty}_{t=0}$ depends on $\{g_t\}^{\infty}_{t=0}$ and k_0, but not on the sequence of lump-sum taxes, $\{T_t\}^{\infty}_{t=0}$. While uniquely, the optimal value of b_{t+1} depends on $\{T_t\}^{\infty}_{t=0}$. It also depends on $\{g_t\}^{\infty}_{t=0}$ (given its the government's selling of bonds, it all checks out).

Finally, we can use the government budget identity and solvency condition to **solve for the government's intertemporal budget constraint**. We first start by iterating the budget constraint (given) in order to get the present value of the constraint to hold over time. Starting with the budget:

$$\begin{aligned}
b_{t+1} &= (1 + r_t)b_t - T_t + g_t \rightharpoonup \text{ starting point} \\
b_1 &= (1 + r_0)b_0 - T_0 + g_0 \\
b_2 &= (1 + r_1)b_1 - T_1 + g_1 \rightarrow \text{ plug in } b_1 \rightarrow \\
b_2 &= (1 + r_1)((1 + r_0)b_0 - T_0 + g_0) - T_1 + g_1 \rightarrow
\end{aligned}$$

$$b_{t+1} = (1 + r_0)\sum_{t=1}^{\infty} R_t^{-1} b_0 - \sum_{s=t+1}^{\infty} R_s^{-1}(T_s - g_s)$$

now multiply both sides by R_t:

$$\sum_{t=1}^{\infty} R_t^{-1} b_{t+1} = (1 + r_0)b_0 - \sum_{t=1}^{\infty} R_t^{-1}(T_t - g_t)$$

Imposing the solvency constraint, we know that $\sum_{t=1}^{\infty} R_t^{-1} b_{t+1} \rightarrow \lim_{t\to\infty} R_t^{-1} b_{t+1} = 0$ in a competitive equilibrium. Therefore, we rearrange and get:

$$(1 + r_0)b_0 = \sum_{t=1}^{\infty} R_t^{-1}(T_t - g_t) + \lim_{t\to\infty} R_t^{-1} b_{t+1}$$

where in the limit, the final term, $\lim_{t\to\infty} R_t^{-1} b_{t+1}$ will reduce to zero. $(1 + r_0)b_0$ is the amount (debt) owed at the end of period 0. This intertemporal budget constraint tells us that the present value of future surpluses equals the total payoff due on existing debt at the end of period 0. The government budget constraint is the only restriction on the path of taxes, so that the timing of taxes has no effect on household utility or the growth of the economy.

[11] The given functional form in this problem is $f(k_t) = y_t$. Because n_t in this problem is normalized to 1, the $\frac{1}{n_t}$ term that appears in the first derivative wrt k_t will just $= 1$, meaning that it does not disrupt or change the results of our explanation above. As we will see in the case of *distortionary taxes*, this condition may change. In the case of lump sum, this holds.

[12] Note the resulting formula with k_t comes directly from subtracting b_{t+1} from a_{t+1}.

4.2.3 Distortionary taxation

Solving for the competitive equilibrum:

We start with the example[13] that allows the government to impose distortionary taxes. We start by solving for the competitive equilibrium, **using a decentralized, and not a ramsey approach**, ensuring the feasibility condition[14] is satisfied. Labor (τ_t^n) and capital (τ_t^k) income taxes are possible:

Step 1: Set up household's optimization problem and make necessary assumptions. Below is the general form of the Lagrange, before taxes are added:

$$\max_{\{c_t,n_t,a_{t+1}\}} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t(u(c_t) - v(n_t)) - \lambda_t(a_{t+1} - (1+r_t)a_t + (1-\tau_t^n)w_t n_t + c_t)$$

Now, a second option is to add taxes and expand $a_{t+1} = k_{t+1} + b_{t+1}$[15] such that:

$$\max_{\{c_t,n_t,k_{t+1},b_{t+1}\}} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t(u(c_t) - v(n_t)) - \lambda_t(k_{t+1} + b_{t+1} - (1+r_t)b_t - ((1+(1-\tau_t^k)(\nu_t - \delta))k_t + c_t - (1-\tau_t^n)w_t n_t)$$

We start by using the standard approach, listing out the assumptions for the *household* we need:

- $0 \leq n_t \leq 1$: constraint on labor
- $a_{t+1} - (1+r_t)a_t + c_t - (1-\tau_t^n)w_t n_t$: household budget identity
- $R_t^{-1} a_{t+1} \geq 0$: household asset solvency constraint
- a_0: given

Then we use the standard approach, such that we optimize with respect to $\{c_t, n_t, a_{t+1}\}_{t=0}^{\infty}$:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= 0 = \beta^t(u'(c_t) - \lambda_t) && \implies \lambda_t = u'(c_t) \\
\frac{\partial \mathcal{L}}{\partial n_t} &= 0 = \beta^t(v'(n_t) - \lambda_t((1-\tau_t^n)w_t) && \implies v'(n_t) = u'(c_t)(1-\tau_t^n)w_t \\
\frac{\partial \mathcal{L}}{\partial a_{t+1}} &= 0 = -\beta^t \lambda_t + \beta^{t+1}\lambda_{t+1}(1+r_{t+1}) && \implies \frac{u'(c_t)}{u'(c_{t+1})} = \beta(1+r_{t+1}) \\
\frac{\partial \mathcal{L}}{\partial k_t} &= 0 = -\beta^t \lambda_t + \beta^{t+1}((1-\tau_{t+1}^k)(\nu_{t+1} - \delta)) && \implies \frac{u'(c_t)}{u'(c_{t+1})} = \beta((1-\tau_{t+1}^k)(\nu_{t+1} - \delta))
\end{aligned}$$

From the first order conditions, we get the four necessary conditions for the household's optimality problem:

1. $\dfrac{u'(c_t)}{u'(c_{t+1})} = \beta(1+r_{t+1})$: euler equation
2. $v'(n_t) = u'(c_t)(1-\tau_t^n)w_t$: labor-leisure tradeoff
3. $a_{t+1} = (1+r_t)a_t - c_t + (1-\tau_t^n)w_t n_t$: household budget identity holds
4. $\lim\limits_{t\to\infty} R_t^{-1} u'(c_t) a_{t+1} = 0$: transversality condition(s) holds

Step 2: Now we want to optimize for firms in a decentralized equilibrium by maximizing profits. Firms rent capital and hire labor, so the optimization problem is:

$$\max_{n_t,k_t} \{f(k_t,n_t) - w_t n_t^d - \nu_t k_t^d\}$$

[13] Taken from the Spring 2023 macro preliminary exam. Note that the spring 2025 preliminary exam is very similar in the problem structure.

[14] The feasibility condition ensures the FOCs in the competitive equilibrium are sufficient, by adhering to the resource constraint in the wider economy. Productive output, $y_t = f(k_t, n_t)$ must be equivalent to the sum of economy wide spending, c_t, i_t, g_t. Feasibility is necessary in both competitive equilibrium and ramsey primal approach. In a competitive equilibrium, even though government expenditures, $\{g_t\}_{t=0}^{\infty}$ do not appear in the household's budget constraint is satisfied by ensuring the feasibility conditions are met.

[15] It's only possible to do this if you know the constraint on capital, such that $\delta + r_t = (1-\tau_t^k)\nu_t$. Where ν_t is the rental rate of capital. Here we know this must hold in equilibrium because firms will be taxed on the value of the capital, the rents accrued during the period's rental, less the depreciation in the value.

where $f(k_t, n_t) = y_t$ is the production function, which must, in equilibrium sum to $c_t + i_t + g_t$. The first order conditions from profit maximization will give us the factor market clearing conditions, seen in step 3.

Step 3: Markets clear. Market clearing conditions are found from the first order conditions, such that:

1. Asset markets clear: $a_{t+1} = k_{t+1} + b_{t+1}$
2. Goods market clear[16]: $k_{t+1} = (1 - \tau_t^n) w_t n_t + (1 - \tau_t^k)(\nu_t - \delta) k_t - c_t - g_t$
3. Factor markets clear: $n_t = n_t^d$ and $k_t = k_t^d$, with equilibrium prices:
   - $\nu_t = f_k'(\cdot)$: rental rate of capital
   - $w_t = f_n'(\cdot)$: wages

Step 4: We need to find the government's budget constraint and that it is satisfied. Recall that in general, the government is constrained by total production in the economy, such that $y_t = g_t + i_t + c_t$. We know the following equivalencies hold for the resource identity that results from taking the difference between the household and government budget identities:

$$\begin{aligned} k_{t+1} =& f(k_t, n_t) + (1-\delta) k_t - c_t - g_t \\ k_{t+1} =& \tau_t^n w_t n_t + (1 + (1 - \tau_t^k)(\nu_t)) k_t - c_t - g_t \end{aligned}$$

Then, by subtracting from the household budget identity

$$\begin{aligned} k_{t+1} + b_{t+1} &= (1 + r_t) b_t - (1 - \tau_t^n) w_t n_t - (1 - (1 - \tau_t^k)(\nu_t - \delta)) k_t - c_t \\ -k_{t+1} &= (\tau_t^n) w_t n_t + (1 + (1 - \tau_t^k)(\nu_t)) k_t - c_t - g_t \\ b_{t+1} &= (1 + r_t) b_t + \tau_t^k \nu_t (k_t) + \tau_t^n (w_t n_t) - g_t \end{aligned}$$

we can see government's constraint, where the budget identity is:

$$b_{t+1} = (1 + r_t) b_t + \tau_t^k \nu_t (k_t) + \tau_t^n (w_t n_t) - g_t$$

It is possible to stop here in the problem, as we display the final version of the government's budget constraint. But we can also iterate it forward so that it holds for all time periods *(it becomes intertemporal).* So we do the following:

$$\begin{aligned} b_{t+1} &= (1 + r_t) b_t + \tau_t^k \nu_t (k_t) + \tau_t^n (w_t n_t) - g_t \\ b_1 &= (1 + r_0) b_0 + \tau_0^k \nu_0 (k_0) + \tau_0^n (w_0 n_0) - g_0 \\ b_{t+1} &= (1 + r_t)[(1 + r_0) b_0 + \tau_0^k \nu_0 (k_0) + \tau_0^n (w_0 n_0) - g_0] + \tau_t^k \nu_t (k_t) + \tau_t^n (w_t n_t) - g_t \end{aligned}$$

Where the next step is to combine terms and rearrange to isolate government bonds, and take the product of the discount and each term for $t \to \infty$:

$$\begin{aligned} (1 + r_t) b_{t+1} =& (1 + r_0) b_0 + (1 + r_t)[\tau_0^k \nu_0 (k_0) + \tau_0^n (w_0 n_0) - g_0] \\ &+ \tau_t^k \nu_t (k_t) + \tau_t^n (w_t n_t) - g_t \\ \Pi_{s=1}^t (1 + r_s) b_{t+1} =& \Pi_{s=1}^t (1 + r_0) b_0 + \Pi_{s=1}^t (1 + r_t)[\tau_0^k \nu_0 (k_0) + \tau_t^k \nu_t (k_t)] \\ &+ \Pi_{s=1}^t (1 + r_t)[\tau_0^n (w_0 n_0) + \tau_t^n (w_t n_t)] - \Pi_{s=1}^t (1 + r_t)[g_0 + g_t] \\ R_t^{-1} b_{t+1} =& R_0^{-1} (1 + r_0) b_0 + R_t^{-1}[\tau_t^k \nu_t (k_t)] + R_t^{-1}[\tau_t^n (w_t n_t)] - R_t^{-1} g_t \end{aligned}$$

Now impose the solvency condition[17] on government debt, b_t, and rearrange:

$$\begin{aligned} \lim_{t \to \infty} R_t^{-1} b_{t+1} &= (1 + r_0) b_0 + R_t^{-1}[\tau_t^k \nu_t (k_t)] + R_t^{-1}[\tau_t^n (w_t n_t)] - R_t^{-1} g_t \\ -(1 + r_0) b_0 &= R_t^{-1}[g_t + \tau_t^k \nu_t (k_t) + \tau_t^n (w_t n_t)] + \lim_{t \to \infty} R_t^{-1} b_{t+1} \\ (1 + r_0) b_0 &= \sum_{t=0}^{\infty} R_t^{-1}[\tau_t^k \nu_t (k_t) + \tau_t^n (w_t n_t)] - \sum_{t=0}^{\infty} R_t^{-1} g_t \end{aligned}$$

[16] Where, in the most general form, we use $k_{t+1} = f(k_t, n_t) + (1 - \delta) k_t - c_t - g_t$, or the economy wide resource identity, showing how capital flows with the inclusion of the public expenditure

[17] Where $\lim_{t \to \infty} R_t^{-1} b_{t+1} = 0$ because, alongside the TVC conditions we know that the stock of debt must converge to zero in the limit.

The key takeaway from the government's intertemporal budget is that the government debt is negative (i.e. the government is liquid), such that the sum of discounted debt and the sum of present value spending is less than or equal to transfers, or in this case, spending less tax revenue. Above we see the present value of debt is equal to discounted consumption less taxable amount of 'productivity' in the economy.

With all of these conditions, we show that the necessary conditions are sufficient, and as thus, the competitive equilibrium is both optimal, and satisfies the government's budget constraint for a given fiscal policy, $\{g_t, \tau_t^n, \tau_t^c, \tau_t^k\}_{t=0}^{\infty}$.

Ramsey approach:

We define a ramsey equilibrium as an allocation such that the social planner optimizes the household utility, taking prices as given, where the government's budget constraint is satisfied. In a ramsey equilibrium, a competitive equilibrium is achieved, such that markets clear.

Steps 1-4 as listed are still relevant, although slightly augmented. We will still need the FOCs from the household, and the same constraints, although the key difference is in our primal approach, where we write the implementability condition using household budget identity, a_0, the TVCs, and the FOCs.[18]

Step 1: The social planner maximizes household utility

$$\max_{\{c_t, n_t, a_{t+1}\}} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t (u(c_t) - v(n_t)) - \lambda_t (a_{t+1} - (1+r_t)a_t + c_t - (1-\tau_t^n)w_t n_t - (1-\tau_t^k)(\nu_t - \delta)k_t)$$

subjected to:

- the budget identity (listed in the optimization problem)
- the resource identity: $k_{t+1} = f(k_{t+1}, n_{t+1}) + (1-\delta)k_t - g_t - c_t$
- euler equation
- labor-leisure trade-off
- prices (as given)
- solvency
- initial conditions, $k_{t+1} \geq 0, b_{t+1} \geq 0$, a_0 is given

Step 2: The social planner must form the implementability condition. Begin by first multiplying both sides by the pricing kernel, or the value of $\lambda_t = u'(c_t)$ from the FOCs. Then, we will substitute the household's other FOCs into the intertemporal budget constraint:

$$\begin{aligned}
\lambda_t * ((1+r_0)a_0) &= \left(\sum_{t=0}^{\infty} R_t^{-1} c_t - \sum_{t=0}^{\infty} R_t^{-1} [(1-\tau_t^n)(w_t n_t)] + \left[(1 + (1-\tau_t^k)(\nu_t - \delta)k_{t+1}\right] \right) * \lambda_t \\
(1+r_0)(b_0 + k_0)u'(c_t) &= \sum_{t=0}^{\infty} \beta^t \frac{u'(c_{t+1})}{u'(c_t)} c_t u'(c_t) - \sum_{t=0}^{\infty} R_t^{-1} [(1-\tau_t^n)(w_t n_t) u'(c_t)] + \sum_{t=0}^{\infty} R_t^{-1} k_{t+1} \\
(1+r_0)(b_0 + k_0)u'(c_t) &= \sum_{t=0}^{\infty} \beta^t u'(c_{t+1}) c_t + \sum_{t=0}^{\infty} R_t^{-1} v'(n_t) n_t + \lim_{t \to \infty} \mathit{\Pi} R_t^{-1} k_{t+1} \\
(1+r_0)a_0 u'(c_t) &= (1+r_0)u'(c_0) + \sum_{t=0}^{\infty} \beta^t v'(n_t) n_t \\
(1+r_0)a_0 u'(c_0) &= \sum_{t=0}^{\infty} \beta^t v'(n_t) n_t \\
0 &= \sum_{t=0}^{\infty} \beta^t v'(n_t) n_t - (1+r_0)a_0 u'(c_0)
\end{aligned}$$

[18] Note that the government budget constraint is implied in the implementability condition, and as such we don't need it in the ramsey planner problem. If we were solving directly for the taxation pathways as the government, then we would need it!

Step 3: Now the planner maximizes the household's utility against the resource constraint and the implementability condition.

$$\max_{\{c_t,k_{t+1},n_t\}} \sum_{t=0}^{\infty} \beta^t (u(c_t) - v(n_t)) - \lambda_t \left[k_{t+1} - (1-\delta)k_t - f(k_t, n_t) + g_t + c_t\right] -$$
$$\mu_t \left[\sum_{t=0}^{\infty} \beta^t v'(n_t) n_t - (1+r_0) a_0 u'(c_0)\right]$$

Recall we also know that k_0, b_0 are given, and the problem is subject to the following constraints:

- $k_{t+1} \geq 0$
- $l_t, n_t \geq 0$
- $l_t + n_t = 1$ for all $t \geq 0$

Step 4: Take FOCs from this maximization problem, such that
Step 5: To get the optimal fiscal policy pathway, $\{\tau_t^n, \tau_t^c, \tau_t^k\}$, we must compare the difference between the planner and the household's FOCs. The wedge, or difference between two will show the optimal tax rate for achieving a competitive equilibrium.

4.2.4 Interpretation of effects

- Some key takeaways are to remember that the timing of government spending, g_t does effect the equilibrium allocation and prices.
- Another main concern is the optimal rate of taxation. In the case of capital income tax rate, the Euler equation is satisfied when $\tau_{t+1}^k = 0$ for all $t > 0$. But, the euler conditions hold if $\tau_1^k > 0$ because the present value of the initial asset holdings, a_0 is positive whenever the tax rate is positive if a_0 is not zero to start with. Recall that a_0 can never be zero to start, or else it would look the same as a 100% capital level in $t = 0$.
- Often we are better off with a capital tax, because it doesn't effect productivity, even though it has a uniquely distorting effect on the equilibrium.
- The timing of taxes matter when labor supply is elastic because labor supply changes with tax rate changes. This is due to the permanent labor hypothesis, and thus the elasticities determine whether consumption rises or falls in response to a tax postponement.
- The tax on labor does not effect the next period's labor supply. This means in the next period you still have the same supply, although the tradeoff between periods may change, due to a change in the future budget constraint.
- Another important point in interpretation is that *in the ramsey set up* the maximization function does not include the capital income tax. That is because, if the government was considering this is the objective function, the Kuhn-Tucker conditions would be used to find the optimal rate, and it would imply the capital income tax is zero. this therefore means we rule out capital income tax and instead focus on the government's choice of optimal distorting tax.

4.3 Durable goods model

Our next model differentiates between two types of consumption goods: durables and non-durables. Nondurable goods are a *flow variable*, and durable goods are a *stock variable* which provide a flow of services in each period, depreciating over time. Here we need to establish whether we think durable and nondurables goods are complements or substitutes:

$$\text{if } U_D > 0 \rightarrow \textit{complements}$$

$$\text{if } U_D < 0 \rightarrow \textit{substitutes}$$

First we define all new, relevant variables:

1 c_t: real, non-durable good consumption

2 D_t: stock of durable goods

3 d_t: total investment expenditures

4 $\triangle D_{t+1} = d_t - \delta D_t$: such that durable goods in the next period are a function of total investment, less than the depreciation of the existing stock of goods

5 $\triangle a_{t+1} + c_t + p_t^D d_t = x_t + r_t a_t$: where p_t^D is the price of durable goods, relative to non-durables. So, $p_t^D d_t$ is the total expenditure on durable goods, in non-durable prices.

5 Complete markets model

Critically, this economy is **dynamic endowment economy**, where choices are made in response to an exogenous stream of resources (endowments), and there is no production. Agents optimize, and the endowments are determined by the states. With complete markets, we must have market clearing and we therefore always solve for a competitive equilibrium allocation.

5.1 Lucas tree model

The Lucas tree model includes asset pricing. This model is a departure from the previous models we covered in the course, in that it is a model that is *not autarky* and there is *no representative agent*. This model allows for trade, and each agent has their own utility function. Additionally, it incorporates a state space which consist of probability distributions which are associated with payoffs or expected returns in each state. The model will need to achieve *pareto optimality*, there are prices, and there is an asset market for which pricing must also hold.

There are **Arrow-Debreu prices** in the Lucas tree model, such that any type of good can be traded across all time periods and there are complete markets. With this introduction, we can trade in any states for any period, t such that the current state we are is t or $t-n$. By having complete markets with prices, we are able to share *risk* by purchasing contracts on other people's assets. This means risk is pooled, and there are no changes in idiosyncratic risk, only aggregate risk. There are two types of assets: (1) equity claims on the trees and (2) bonds.

5.1.1 Model set up

1. **Endowment:** Here we start with the following endowment sequence:

$$\{y_0^i, \{y^i(s_1)\}_{s_1 \in S}, \{y^i(s_2)\}_{s_2 \in S}\}$$

   Endowment is stochastic, it is a random variable that affects both s, the state and c, consumption, or payoff. $s_t \in S$ is a random variable, drawn from the sample space, S which is finite. We assume s_0 is known.

2. **Household utility function**: Each household, i, has a utility function, affected by consumption in each period, t. Consumption in turn depends on the current state and all historical states. The utility function has standard properties:

   - $u(c)$ is increasing
   - $u(c)$ is strictly concave
   - $u(c)$ is continuous and twice differentiable

   The utility function can be written as:

$$U_0^i = \sum_{t=0}^{\infty} \sum_{s^t}^{S} \beta^t \mathit{\Pi}(s^t) \cdot u(c_t^i(s^t))$$

3. **Resource constraint**: Here the resource constraint ensures that endowments are always greater than or equal to consumption across all periods. It's referred to as the feasible consumption condition. In equilibrium, it holds (in equality). We state:

$$\sum_{i=1}^{I} y^i(s_t) \geq \sum_{i=1}^{I} c_t^i(s^t) \text{ for } t, s^t$$

5.1.2 Pareto problem

We begin by setting up the Pareto problem. Now, we can either solve this directly or we solve for the competitive equilibrium with complete markets. But, we need the necessary conditions from the Pareto problem solve for optimal consumption by each household in each period. Let's begin with the Pareto optimization problem:

$$\overset{max}{\{c_t^i(s^t)\}_{i,t,s^t}} \sum_{i=1}^{I} \lambda_i U_0^i$$

We use the household's utility function as defined above for U_0^i, given that $\lambda_i > 0$ for each i subject to the resource constraint (also defined above). Now we solve the Lagrange:

$$L = max \sum_{i=1}^{I} \sum_{t=0}^{t} \lambda_i \beta^t \pi(s_t) u(c_t^i(s^t)) - \mu_t [\sum_{i=1}^{I} y^i(s^t) - \sum_{i=1}^{I} c^i(s^t)]$$

Now we can derive the first order conditions:

$$\begin{aligned}
\frac{dL}{dc_t^i} &= \beta^t \lambda_t \pi(s^t) u'(c_t^i(s^t)) - \mu_t = 0 \\
\frac{dL}{ds^t} &= \beta^t \lambda_t \pi'(s^t) u(c_t^i(s^t)) c_t^{i'}(s^t) - \mu_t [y^{i'}(s^t) - c^{i'}(s^t)] = 0
\end{aligned}$$

These imply the necessary conditions, where we can take the ratio of $\mu_t(s^t)$ for the households i, j:

$$\frac{\beta^t \pi(s^t) u'(c^i(s^t))}{\beta^t \pi(\hat{s}^t) u'(c^i(s^t))} = \frac{\beta^t \pi(s^t)}{\beta^t \pi(\hat{s}^t)} \cdot \frac{u'(c^j(s^t))}{u'(c^j(s^t))}$$

From this we get the following ratio holds:

$$u'(c^i(s^t)) = \frac{\lambda_i}{\lambda_j} \cdot u'(c^j(s^t)) \beta^t \pi(s^t)$$

This gives us some form of a Euler equation. As a result, there are two conditions we need for the Pareto optimum to hold, which is:

$$\sum_{i=1}^{I} c^i(s^t) = \sum_{i=1}^{I} y^i(s^t) \tag{4}$$

$$\frac{\beta^t \pi(s^t) u'(c^i(s^t))}{u'(c^i(s_0))} = q_t^0(s^t) \tag{5}$$

Equation (5) is the marginal rate of substitution for one good today, in state t, for the same good in terms of the utility derived from the good in the known state, $t = 0$. The new variable introduced, $q_t^0(s^t)$ becomes the price of the good.

5.1.3 Competitive equilibrium

Now we move to finding a competitive equilibrium, where we can maximize each household's utility with prices as given. A competitive equilibrium is an allocation $\{c_t^i(s^t),\ c_t^j(s^t)\}_{t,s^t}$ and prices $\{q_t^0(s^t)\}_{t,s^t}$ such that each household maximizes its utility over its budget, taking prices $\{q_t^0(s^t)\}_{t,s^t}$ as given, and markets clear for all dates and states. At the competitive equilibrium allocation, the necessary conditions for a Pareto optimum must also be met.

This problem is similar to the Pareto problem, where equation (4) becomes the market clearing condition. The resource constraint on households now factors in prices. In this decentralized competitive equilibrium, the household now optimizes against, (a) resource constraint with prices and (b) budget constraint. Households then solve for the first order conditions, and are subjected to (c) the goods market clearing and (d) the asset market clearing. Finally, the (d) solvency and (e) transversality condition must hold in the market equilibrium. Now we start with a simple maximization problem for the *competitive equilibrium*:

$$\underset{\{c_t^i(s^t)\}_{t,s^t}}{max} U_0^i \text{ s.t. } \sum_{t,s^t} q_t^0(s^t) c^i(s^t) \leq \sum_{t,s^t} q_t^0(s^t) y^i(s^t)$$

But, this is simply the maximization subject to the resource constraint. We have the market clearing conditions from (4), but we need to introduce the household's budget constraint. The **budget constraint** now takes into account that each commodity must be harvested at date t, and it is either wasted or consumed. Here a sumple budget constraint tells us:

Trade can only occur in the same market. Households can also buy and sell non-contingent discount bonds in the same time period. Buying any selling can occur in every period, but the price is dependent on endowment (stochastic). The budget constraint, **with bonds,** is therefore:

$$\sum x_k^i(s^{t-1})[y^k(s_t) + V_t^k(s^t)] + B_t^i(s^{t-1}) = B_{t+1}^i(s^t)(1 + r_{t-1}(s^t))^{-1} + \sum [x_k^i(s^t)V_t^k(s^t)] + c_t^i(s^t)$$

Where

- $x_k^i(s^{t-1})$ is the share of tree k owned by the household i at the **beginning** of time period t.
- $x_k^i(s^t)$ is the share of tree k owned by the household i at the **end** of time period t.
- $x_k^i(s^{t-1}) - x_k^i(s^t)$: Taking the difference of the two variables is the net share of trees purchased.
- $V_t^k(s^t)$ is the price of tree k on date t. It is the price to buy or sell a tree in the period.
- $B_t^i(s^{t-1})$ is the quantity of bonds held by a household at the beginning of time period, t
- $B_t^i(s^{t-1})(1 + r_{t+1}(s^t))^{-1}$ is the cost of the amount of bonds held by a household at end of date, t. At $t = 0$, $B_t^i = 0$.

With this information we can rewrite the budget constraint at $t = 0$:

$$y^i(s_0) + V^i(s_0) = (1 + r_1)^{-1}B_1^i(s_0) + \sum \left[x_k^i(s^0)V_t^k(s^0)\right] + c_0^i(s^0)$$

Now we set up the Lagrange multiplier. Here we make an assumption that there is a Cobb-Douglas utility function in the form:

$$U_0^i = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \Pi(s^t)\left[\frac{c_t^i(s^t)^{1-\sigma} - 1}{1-\sigma}\right]$$

with Lagrange:

$$\mathcal{L} = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \Pi(s^t)\left[\frac{c_t^i(s^t)^{1-\sigma} - 1}{1-\sigma}\right] - \mu^i B_t^i(s^{t-1}) - \mu^i B_{t+1}^i(s^t)(1 + r_{t+1})^t$$
$$- \mu^i\left[\sum x_k^i(s^{t-1})[y^k(s^t) + V_t^k(s^t)] - \sum x_k^i(s^t)(V_t^k(s^t)) + c_i(s^t)\right]$$

With key restrictions, including:

- $0 \leq x_k^i(s^t) \leq 1 \rightarrow$ implying that equities and bond sales have certain restrictions
- Solvency constraint:
$$\lim_{t\to\infty} \sum \Pi(s^t|s^\tau) \cdot R_{\tau,t}^{-1}(s^t|s^\tau) \cdot B_{t+1}^i(s^t) \geq 0$$
Solvency implies, for all $\tau \geq 0$, no bonds or equities are held indefinitely.
- Recall the market equilibrium:
$$\sum_{i=1}^{I} y^i(s_t) = \sum_{i=1}^{I} c^i(s^t) \quad \forall t, s^t$$
We can rewrite as:
$$c_t^i(s^t) = \gamma^i y(s_t) \quad \text{s.t. } \gamma_i \geq 0 \quad i = 1, ...N$$
where γ is the share of aggregate harvest for t, such that $\sum_{i=1}^{I} \gamma_i = 1$

6 Appendix

6.1 Linearization

Below is a step-by-step approach to linearizing the optimal solution of an infinite horizon model, in order to obtain the values for which the solution is stable at the steady state.

Linearizing Equations of Motion

We begin from the discrete-time system derived by maximizing the planner's utility:

$$\begin{aligned} u'(c_t) &= (1-\delta+f'(k_{t+1}))\beta u'(c_{t+1}) \\ k_{t+1}-k_t &= f(k_t)-\delta k_t - c_t \end{aligned}$$

1. Derive the steady state conditions (from the Lagrange):

$$\begin{aligned} c^* &= f(k^*)-\delta k^* \\ f'(k^*) &= \delta+\rho, \quad \text{where } \rho = \frac{1}{\beta}-1 \end{aligned}$$

2. Now we want to linearize the two steady state conditions (Euler and law of capital motion) using Taylor expansions around (c^*, k^*). Recall this requires defining

$$dc_t \approx (c_t - c^*)$$

$$\begin{aligned} u''(c^*)(dc_t) &= (1-\delta+f'(k^*))\beta u''(c^*)dc_{t+1} + f''(k^*)\beta u'(c^*)dk_{t+1} \\ &\Longrightarrow \\ u''(c^*)(c_t-c^*) &= (1-\delta+f'(k^*))\beta u''(c^*)(c_{t+1}-c^*) + f''(k^*)\beta u'(c^*)(k_{t+1}-k^*) \end{aligned}$$

$$\begin{aligned} dk_{t+1}-dk_t &= (f'(k^*)-\delta)dk_t - dc_t \\ &\Longrightarrow \\ (k_{t+1}-k^*)-(k_t-k^*) &= (f'(k^*)-\delta)(k_t-k^*)-(c_t-c^*) \end{aligned}$$

3. Rearrange the conditions to simplified versions:

$$\begin{aligned} (1) \quad u''(c^*)\left[(c_t-c^*)-(1-\delta+f'(k^*))\beta(c_{t+1}-c^*)\right] &= f''(k^*)\beta u'(c^*)(k_{t+1}-k^*) \\ (2) \quad k_{t+1}-k_t &= \rho(k_t-k^*)-(c_t-c^*) \end{aligned}$$

4. Now we want to rewrite the Euler equation by plugging in the linearized resource constraint (2) into the RHS of the equation. *Note:* $(\beta(1-\delta)+f'(k^*))=1$, this is important for getting the below equations. The purpose of this is to eliminate dk_{t+1} from the system of equations:

$$-u''(c^*)(c_{t+1}-c_t) = f''(k^*)u'(c^*)(k_t-k^*) - f''(k^*)\beta u'(c^*)(c_t-c^*)$$

5. Then, we can further rearrange to isolate the variables of interest, which is year over year changes in the optimal values. These variables are:

$$\begin{bmatrix} c_{t+1}-c_t \\ k_{t+1}-k_t \end{bmatrix}$$

6. So we isolate these variables on the RHS for the Euler equation. The resource identity is already in form. This gives:

$$\begin{aligned} c_{t+1}-c_t &= \frac{u'(c^*)}{-u''(c^*)f''(k^*)}(k_t-k^*) - \beta\frac{u'(c^*)}{-u''(c^*)f''(k^*)}(c_t-c^*) \\ k_{t+1}-k_t &= \rho(k_t-k^*)-(c_t-c^*) \end{aligned}$$

7. Lets define the following variable: $\sigma(c^*) \equiv -\frac{c^*u''(c^*)}{u'(c^*)}$. From this we can now rewrite the Euler equation by subbing in the σ

$$c_{t+1}-c_t = \frac{c^*}{\sigma(c^*)f''(k^*)}(k_t-k^*) - \beta\frac{c^*}{\sigma(c^*)f''(k^*)}(c_t-c^*)$$

8. Now, we put it all together in matrix form (as seen in 3.5).

$$\begin{bmatrix} c_{t+1} - c_t \\ k_{t+1} - k_t \end{bmatrix} = \begin{bmatrix} -\beta \dfrac{c^*}{\sigma(c^*) f''(k^*)} & \dfrac{c^*}{\sigma(c^*) f''(k^*)} \\ -1 & \rho \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}$$

9. Solve for eigenvalues, λ:

$$\det(A - \lambda I) = 0 \Rightarrow \lambda^2 + \left(\beta \frac{c^*}{\sigma(c^*) f''(k^*)} - \rho \right) \lambda + \beta \frac{c^*}{\sigma(c^*) f''(k^*)} = 0$$

We use the quadratic formula to get:

$$\lambda_{\pm} = \frac{1}{2} \left[- \left(\frac{\beta}{\sigma(c^*)} c^* f''(k^*) - \rho \right) \pm \left(\left(\frac{\beta}{\sigma(c^*)} c^* f''(k^*) - \rho \right)^2 - 4 \frac{\beta}{\sigma(c^*)} c^* f''(k^*) \right)^{\frac{1}{2}} \right]$$

Because $f''(k^*) < 0$ and $\rho > 0$, the eigenvalues satisfy

$$\lambda_- < 0 \quad \text{and} \quad \lambda_+ > \rho - \frac{\beta}{\sigma(c^*)} c^* f''(k^*) > \rho > 0.$$

The eigenvectors for the linearized system are found by using

$$\lambda \nu = A \nu$$

So what is ν? It is a column vector that represents the eigenvector, where they are linearly independent, such that

$$\nu = \begin{bmatrix} m \\ 1 \end{bmatrix} \implies \nu_- = \begin{bmatrix} c_1 \\ k_0 \end{bmatrix}, \quad \nu_+ = \begin{bmatrix} c_t \\ k_0 \end{bmatrix}$$

And we can plug back into the key formula, $A\nu = \lambda\nu$, which is the formula for diagonalization. From the following we will get the signs and slopes of the eigenvectors:

$$[\nu_- \quad \nu_+] \cdot \begin{bmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{bmatrix} = A \cdot [\nu_- \quad \nu_+]$$

10. The following step is to put together the two necessary conditions and the transversality condition to get a generalized solution for the linearized system: The general solution for the linearized system is given by

$$\begin{bmatrix} c_{t+1} - c^* \\ k_{t+1} - k^* \end{bmatrix} = \begin{bmatrix} 1 - \beta \dfrac{c^*}{\sigma(c^*)} f''(k^*) & \dfrac{c^*}{\sigma(c^*)} f''(k^*) \\ -1 & \dfrac{1}{1+\rho} \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix},$$

when we write it in terms of $c_{t+1} - c^*$ and $k_{t+1} - k^*$ on the left-hand side. Using the eigenvectors and eigenvalues, we can express any path as

$$\begin{bmatrix} c_{t+1} - c^* \\ k_{t+1} - k^* \end{bmatrix} = [\alpha \nu_- (1 + \lambda_-) + (1 - \alpha) \nu_+ (1 + \lambda_+)](k_t - k^*)$$

where

$$[\alpha \nu_- + (1 - \alpha) \nu_+](k_t - k^*) = \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}.$$

Using the above definition for $\nu = \begin{bmatrix} m \\ 1 \end{bmatrix}$, we can see that $\dfrac{c_t - c^*}{k_t - k^*}$ determines the parameter α.

Now, iterate this back to zero to get

$$\begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix} = [\alpha \nu_- (1 + \lambda_-)^t + (1 - \alpha) \nu_+ (1 + \lambda_+)^t](k_0 - k^*)$$

where c_0 is the only undetermined variable, which we can find from applying the transversality condition. This condition is essential where we can see when you substitute in the linearized system, the condition needs to hold.

11. Consider two cases: $\alpha = 1$ and $\alpha = 0$.

(a) If $\alpha = 1$, we have that

$$c_t - c^* = (1+\lambda_-)^t m_-(k_0 - k^*) \quad \text{and} \quad k_t - k^* = (1+\lambda_-)^t (k_0 - k^*).$$

To demonstrate that these solutions satisfy the transversality condition, we can use a first-order approximation writing

$$u'(c_T) = u'(c^*) + u''(c^*)(c_t - c^*)$$

and substitute into the transversality condition,

$$\begin{aligned}\lim_{T\to\infty} \beta^T [u'(c^*) + u''(c^*)(c_t - c^*)][k^* + (1+\lambda_-)^t(k_0 - k^*)] = \\ \lim_{T\to\infty} \beta^T u'(c^*)[k^* + (1+\lambda_-)^t(k_0 - k^*)] + \\ \lim_{T\to\infty} \beta^T u''(c^*)(c_t - c^*)[k^* + (1+\lambda_-)^t(k_0 - k^*)].\end{aligned}$$

Observe that $\lim_{T\to\infty} \beta^T u'(c^*)[k^* + (1+\lambda_-)^t(k_0 - k^*)] = 0$ and

$$\lim_{T\to\infty} \beta^T u''(c^*)(c_t - c^*)[k^* + (1+\lambda_-)^t(k_0 - k^*)] =$$

$$\lim_{T\to\infty} \beta^T u''(c^*)(1+\lambda_-)^t m_-(k_0 - k^*)[k^* + (1+\lambda_-)^t(k_0 - k^*)] = 0$$

since $0 < \beta < 1$ and $\lambda_- < 0$.

(b) Next, if $\alpha = 0$,

$$c_t - c^* = (1+\lambda_+)^t m_+(k_0 - k^*) \quad \text{and} \quad k_t - k^* = (1+\lambda_+)^t (k_0 - k^*).$$

Since $\lambda_+ > \rho$, $(1+\lambda_+)\beta > 1$, the

$$\lim_{T\to\infty} \beta^T u'(c^*) \left[k^* + (1+\lambda_+)^t(k_0 - k^*)\right] = \infty.$$

The transversality condition will not be satisfied for any α not equal to one.

The optimal solution is the unique solution given by

$$c_t - c^* = (1+\lambda_-)^t m_-(k_0 - k^*) \quad \text{and} \quad k_t - k^* = (1+\lambda_-)^t (k_0 - k^*).$$

This means that $c_0 - c^* = m_-(k_0 - k^*)$ solves for c_0 and the optimal growth path follows the stable saddle path towards the steady state. It converges asymptotically from (k_0, c_0) to (c^*, k^*).

A special case makes checking the transversality condition clearer. Let $\sigma > 0$ be constant. Then we do not need to approximate marginal utility, $u'(c_T)$. The transversality condition is

$$\lim_{T\to\infty} \beta^T c_T^{-\sigma} k_{T+1} = 0.$$

Substitute in the solution for $\alpha = 1$,

$$c_t - c^* = (1+\lambda_-)^t m_-(k_0 - k^*) \quad \text{and} \quad k_t - k^* = (1+\lambda_-)^t (k_0 - k^*),$$

to get

$$\lim_{T\to\infty} \beta^T (1+\lambda_-)^{-\sigma t} m_-^{-\sigma}(k_0 - k^*)^{-\sigma} \left[k^* + (1+\lambda_-)^t(k_0 - k^*)\right] = 0.$$

Stable path implies using only λ_-:

$$\begin{aligned} c_t - c^* &= (1+\lambda_-)^t m_-(k_0 - k^*) \\ k_t - k^* &= (1+\lambda_-)^t (k_0 - k^*) \end{aligned}$$

6.2 Euler equations

Model Variant	Euler Equation
standard ramsey growth model	$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]$
with lump sum taxes (τ)	$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]$
with c_t taxes (τ_t^c)	$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{1+\tau_{c,t}}{1+\tau_{c,t+1}} \cdot \beta(1 + r_{t+1})$
with capital taxes (τ_t^k)	$u'(c_t) = \beta u'(c_{t+1})[(1 - \tau_{t+1}^k)f'(k_{t+1}) + 1 - \delta]$
with labor taxes (τ_t^l) [19]	$u'(c_t) = \beta u'(c_{t+1})[r_{t+1} + 1 - \delta]$
Yeomoni peasant	$u'(c_t) = \beta u'(c_{t+1})(1 + r)^t$
with i_t adjustment costs	$u'(c_t) = \beta u'(c_{t+1}) \left[\frac{\partial \phi(I_{t+1}, K_{t+1})}{\partial I_{t+1}} + 1 - \delta\right]$
Lucas tree model (asset pricing)	$1 = \mathbb{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} R_{t+1}\right]$
with government debt (B_t) *(subject to solvency)*	$u'(c_t) = \beta u'(c_{t+1}) \left[r_{t+1} + 1 - \delta\right]$

Table 1: Euler equations across neoclassical model variants

6.3 Budget constraints

Model Variant	Household Budget Constraint
standard Ramsey growth model	$k_{t+1} = (1 - \delta)k_t + i_t - c_t$
with lump sum taxes (τ_t)	For all **fiscal policy problems**, assume $a_{t+1} = k_{t+1} + b_{t+1}$ holds $k_{t+1} + b_{t+1} = (1 + r_t)b_t + (1 + r_t - \delta)k_t + w_t n_t - c_t - \tau_t$
with c_t taxes (τ_t^c)	$k_{t+1} + b_{t+1} = (1 + r_t)b_t + (1 + r_t - \delta)k_t + w_t n_t - (1 + \tau_t^c)c_t$
with capital taxes (τ_t^k)	$k_{t+1} + b_{t+1} = (1 + r_t)b_t + \left[1 + (1 - \tau_t^k)(\nu_t - \delta)\right] k_t + w_t n_t - c_t$ where market return is $r_{t+1} = (1 - \tau_{t+1}^k)(\nu_{t+1} - \delta)$
with labor taxes (τ_t^l)	$k_{t+1} + b_{t+1} = (1 + r_t)b_t + (1 + r_t - \delta)k_t + (1 - \tau_t^l)w_t n_t - c_t$
Yeomoni peasant	simple: $a_{t+1} = (1 + r_t)a_t + w_t n_t - c_t$ intertemporal: $a_{t+1}(\frac{1}{1+r})^t = (1 + r_0)a_0 + \sum_{s=0}^t w_s n_s (\frac{1}{1+r})^s - \sum_{s=0}^t c_s (\frac{1}{1+r})^s$
with i_t adjustment costs	$a_{t+1} = (1 + r_t)a_t + w_t n_t - c_t - \phi(i_t, k_t)$
Lucas tree model (asset pricing)	$p_t a_t = \mathbb{E}_t[p_{t+1}a_{t+1} + d_{t+1}a_{t+1}] + c_t$
with government debt (B_t)	$k_{t+1} + b_{t+1} = (1 + r_t)b_t + (1 + r_t - \delta)k_t + w_t n_t - c_t$

Table 2: Household budget constraints across neoclassical model variants