Intermediate Microeconomics (ECON100A)

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Midterm 1 Review

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[1] *Notes and content were adapted from 2024-25 UCSC Economics PhD ECON204A-C notes, Hal Varian's Microeconomics textbook*

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Contents

1 Math Review

required/requested materials

- **Commutative laws**
  - $A + B = B + A$
- **Distributive laws**
  - $A(B + C) = AB + AC$
  - $(A + B)C = AC + BC$
  - For multiplication, $ab = ba$ but for matrices it is **not** true that $AB = BA$
- **Associative laws**
  - $(A + B) + C = A + (B + C)$
  - $(AB)C = A(BC)$
- **Transpose**
  - Obtained by interchanging rows and columns: $A_{ij} \to A_{ji} = A^T$
  - Rules for the transpose:
    * $(A + B)^T = A^T + B^T$
    * $(A - B)^T = A^T - B^T$
    * $(A^T)^T = A$
    * $(rA)^T = rA^T$
    * $(AB)^T = B^T A^T$ (proof required)
- **Special matrices**
  - Square ($k = n$, same number of rows and columns)
  - Diagonal ($k = n$, and $a_{ij} = 0$ where $i \neq j$)
  - Identity: ones on the diagonal, and zeros elsewhere (i.e. $a_{ij} = 1$ where $a_{ij} = a_{jj}$)
  - Lower (and upper) triangular matrices: all entries below (or above) the diagonal are equal to zero ($a_{ij} = 0$ for all $a_{ij} \neq a_{jj}$)
  - Symmetric: $A^T = A$, i.e. $a_{ij} = a_{ji}$ for all i, j ($|A| = |A^T|$)
  - Idempotent: a square matrix B for which $B^2 = B$ (such as $B = I$)
  - Nonsingular matrix: a square matrix whose rank equals the number of its rows (or columns). If this is a matrix of coefficients in a system of linear equations, the system has *one and only one* solution.
- **Determinants**
  - Useful for defining if a system of linear equations has a solution (chapters 22, 24), for computing a solution where it exists (chapters 11, 14, 22, 24), and for determining whether a given nonlinear system can be well approximated with a linear one (chapter 13, specifically the implicit function theorem).
  - Useful properties:
    * The determinant does not change when you add a multiple of one row to another within A
    * Determinant changes sign when you interchange two rows of A: $|B| = -|A|$
    * If matrix $B = rA$, then $|B| = r \cdot |A|$
    * If two rows of A are the same, then the determinant is zero
    * If A is lower/upper triangular, the determinant is the product of the principal diagonal
    * If R is in row echelon form (REF), then $|R| = \pm|A|$

- *Example cases:* If you have n linearly dependent vectors of order n, then the determinant is zero. If a square matrix of order n has linearly dependent columns, the inverse doesn't exist ($\det = 0$ means no inverse).
- **Theorem 1**: The determinant is defined as a sum of its minors, $a_{ik} \cdot (-1)^{i+j}$. From this, we can prove that the transpose determinant is the same as the determinant of the original matrix A.
- **Theorem 3**: A square matrix A is nonsingular if and only if $|A| \neq 0$. This is true because in REF, A would need to have all diagonal elements non-zero.
- **Theorem 4**: For square matrices, $|AB| = |A| \cdot |B|$

- **Inverse matrices**
  - Purpose: we use inverse matrices in absence of division. Where the inverse is $BA = I = AB$, we define the inverse of A as A^{-1}.
  - The inverse must be (a) square, (b) have the same dimensions as A.
  - $|A^{-1}A| = |I|$, $|A^{-1}||A| = 1$, where $|A| \neq 0$ (otherwise, the inverse does not exist).
  - If A, B are square invertible matrices:
    * $(A^{-1})^{-1} = A$
    * $(A^T)^{-1} = (A^{-1})^T$
    * AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$
    * For any scalar r, rA is invertible and $(rA)^{-1} = \frac{1}{r}A^{-1}$
    * $A^r A^s = A^{r+s}$
  - **Theorem 5**: A^{-1} is unique. Prove this with the identity matrix or by showing the inverse exists by being $\frac{1}{\det(A)}$ times the transpose of the cofactor of A. Proof by contradiction can also show that if two matrices are both inverses, they must be the same by multiplication properties.

Systems of linear equations, *see an example below*

$$\begin{bmatrix} a_1 + 3a_2 \\ 2a_1 + 7a_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = X\mathbf{a}.$$

This is true in general. If

$$X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & ... & \mathbf{x}_n \end{bmatrix} \text{ and } \mathbf{a}^T = (a_1, a_2, ..., a_n)$$

then the linear combination in (8) is a vector that can be written as

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + ... + a_n\mathbf{x}_n = \textstyle\sum_{i=1}^n a_i\mathbf{x}_i = X\mathbf{a}.$$

Theorem 1.1 (Matrix rank and solution) *If A is an $m \times n$ matrix, then $A\mathbf{x} = \theta$ has a solution for a particular value of θ, if and only if (IFF) θ is in col(A). While $A\mathbf{x} = \theta$ has a solution for every single θ iffthe rank is equal to the number of rows (m).*

- **Consistent:** When a system has a solution (when solution is nonzero – affine subspaces).
- **Homogenous:** $A\mathbf{x} = 0$, meaning that the solution set of a homogenous system with n variables is zero and is a subspace of $\mathbb{R}^n$.
  - **Null space:** This is the subspace of solutions in the homogenous system, i.e., the value of zero (this is a subspace).

Building off Theorem 1.1,

Theorem 1.2 *If $\mathbf{d}$ is a solution in $\mathbb{R}^n$ of the system $A\mathbf{x} = \theta$ ($m \times n$ system of linear equations), then every other solution of $\mathbf{x}$ can be written as $\mathbf{x} = \mathbf{d} + \mathbf{v}$ (where $\mathbf{v}$ is in the nullspace(A)). Proof involves setting the definition of null space and the consistent system $A\mathbf{x} = \theta$ up.*

Theorem 1.3 (Fundamental Theorem of Linear Algebra) *Relates the rank to the dimensions of the null space. We see that* $\dim(Null(A)) = number\ of\ columns - rank(A)$*. If you know how to solve (or which systems have) a solution, then when* $A\mathbf{x} = \theta$*, the solution set will be an affine subspace, where the dimensions are equal to the number of variables (columns) less the rank. This is the same as the FTLA where we see that* $\dim = cols - rank$*.*

Linear functions

- These are just mappings from the real space of a certain dimension to another. The functions preserve the vector space structure. We can also call it a linear transformation.
- **Theorem 1.4 (Linear equations)** *There is a vector* $\mathbf{a}$ *in* $\mathbb{R}^n$ *such that* $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x}$ *for all* $\mathbf{x}$ *within* $\mathbb{R}^n$ *if* f *is a linear function. The proof demonstrates that if* $n = 3$*, and we have the basis of* $\mathbb{R}^3$ *in terms of vectors, then we can take any vector* $\mathbf{x}$*'s* (x_1, x_2, x_3) *and multiply it by the bases to get a linear equation. You can then see that this linear combination can be converted to a transpose of the function times the* $\mathbf{x}$ *vector.*
- **Theorem 1.5 (Solution when functions are linear)** *Similarly shows that if* f *is a linear function, there exists* $f(\mathbf{x}) = A\mathbf{x}$ *for all* $\mathbf{x}$ *values in* $\mathbb{R}^2$*.*

Definition of a derivative

- Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The **derivative of f at a point** $x = a$, denoted $f'(a)$, is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

  provided this limit exists.
- The derivative represents the *instantaneous rate of change* of f at the point a.
- If f is differentiable at each point in its domain, then it is differentiable.
- The differentiability of multivariate functions looks similar, except for h is a vector (normalized) and the second term is not just $f(x)$, but is $f(x) + a^T h$. If f is differentiable here, then we can say the derivative is $\mathbb{R}^n$ onto a_x.
- In the most general case where $f : \mathbb{R}^n \to \mathbb{R}^m$, we can create a very general formula for the differentiability of these larger functions.

Below are the set of rules that you can use to take the first derivatives. Note that these are given with a univariate case (one variable), but you can apply the same rules to partial derivatives in a multivariate case, treating the other variable as a constant.

Rule	**Formula**
Constant Rule	$\frac{d}{dx}[c] = 0$
Power Rule	$\frac{d}{dx}[x^n] = nx^{n-1}$
Constant Multiple Rule	$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$
Sum Rule	$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
Difference Rule	$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Optimization:
Constrained optimization is essential to economics. For more information, review UChicago 15 page review on constrained optimization, accessible here `https://home.uchicago.edu/~vlima/courses/econ201/pricetext/chapter2.pdf`.

Definition 1.1 (Constrained optimization) *Constrained optimization studies maxima or minima of a function subject to restrictions on the variables. Examples include portfolio choice with a budget constraint, aircraft design limited by cost or weight, or a hiker whose path is constrained to a trail.*

Formally, the problem is

$$\max_{x \in \mathbb{R}^n} f(x) \quad s.t. \quad g_i(x) = 0 \ (i = 1, \dots, m), \quad h_j(x) \leq 0 \ (j = 1, \dots, p).$$

Equality and inequality constraints reduce the feasible set. Instead of eliminating variables, the method of Lagrange multipliers reformulates the problem by adding multipliers so that first-order conditions describe the optimum.[2]

Theorem 1.6 *Suppose that k is an arbitrary constant and that f and g are differentiable functions at $x = x_0$. Then,*

$$\begin{aligned}
&(a) \quad (f \pm g)'(x_0) = f'(x_0) \pm g'(x_0), \\
&(b) \quad (kf)'(x_0) = k\big(f'(x_0)\big), \\
&(c) \quad (f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0), \\
&(d) \quad \left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}, \\
&(e) \quad \big((f(x))^n\big)' = n(f(x))^{n-1} \cdot f'(x), \\
&(f) \quad (x^k)' = kx^{k-1}.
\end{aligned}$$

Theorem 1.2: *local minimum and maximum conditions*

(a) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a maximum of f;

(b) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a minimum of f; and

(c) If $f'(x_0) = 0$ and $f''(x_0) = 0$, then x_0 can be a maximum, a minimum, or neither.

[2] Adapted from the definition provided by `https://math.gmu.edu/~rsachs/math215/textbook/Math215Ch3Sec7.pdf`

2 Consumer Problem

Chapters 2, 3, and 4 (Slide 1), take a look at `https://www.econgraphs.org/topics/consumer/` for interactive graphs on these topics (and more explanations!). Visualization can be useful.

What is the consumer's problem?

A consumer faces the decision problem of choosing consumption levels for goods and services available for purchase. They must select the best combination of goods that she can afford, as defined by the budget set.

Simple modelling approach

- Two goods: 1, 2
- Two quantities of that good: x_1, x_2
- Prices: p_1, p_2
- Income (total): m
- Budget set: $p_1x_1 + p_2x_2 \leq m$ *(because you cannot buy more of goods (at their prices) then the total income you have to spend).*
- Slope of the budget: $\frac{-p_1}{p_2}$

Definition 2.1 (Goods) *We assume that goods and services are homogeneous commodities, with a finite number of them. The commodity vector (or consumption bundle) is denoted by $x = [x_1, \ldots, x_L]'$, where each element represents a quantity of a specific good in $\mathbb{R}^L$, the commodity space.*

There are certain types of goods, which describe how a consumer sees these different goods, in relation to their income, their preference of the good in relation to other goods, or with respect to price. Below are some *types of goods* that can help explain the way preferences for goods will change over time. These classifications are not always mutually exclusive.

- **Neutral good**: A good in which the consumer is indifferent towards, regardless of the quantity. This implies that a neutral good sits on a vertical indifference curve. A neutral good can be a good (desire more, like another pair of cool sneakers) or a bad (desire less, like pollution).
- **Discrete good**: A good that is only available to be purchased in integer amounts (i.e. you can only buy one car, you cannot buy a part, or a fraction of the car).
- **Giffen good**: Goods where $\frac{dx}{dp} \geq 0$. This indicates that when price for the good increases, demand for the good also increases. This may apply to goods which are valuable, or rare, where demand increases with prices.
- **Normal good**: This is a typical good, where demand increases in income, or $\frac{dx}{dy} \geq 0$. As income rises, an individual will demand more of the normal good.
- **Inferior good**: This is a good that is purchases *less* when income increases. Inferior goods are decreasing in income, where $\frac{dx}{dy} \leq 0$. It's important to note you cannot have a utility function dependent on x_i goods, where $i = 1, ...n$ where all goods are inferior. This would violate our assumption that the utility function is strictly increasing and concave.

Definition 2.2 (Consumption set) *A consumption set is a subset of the commodity space $X \subset \mathbb{R}^L$, containing all consumption bundles an individual can feasibly consume given physical constraints.*[3] *X is*

[3] The key distinction from the budget set lies in the phrase "physical constraints." The consumption set ensures non-negative and finite consumption, but does not impose any "economic constraint".

a set of alternatives or complete consumption plans (universe of alternative choices possible possible by consumer)

- **Consumption bundle** is x where $x \in X$ (vector of $x_1, x_2, \ldots, x_n$ where n is the number of total goods in that bundle).
- Properties of a consumption set: $X \subset \mathbb{R}^n$, X is closed and convex. The null set is always in the set, $0 \in X$.
- **Feasible set:** $B = \{\text{all conceivable and obtainable consumption plans}\} \subseteq X$ regarding the accessing for any constraints. Where $B \subseteq X$.

We assume the set of all non-negative bundles of commodities is represented as:

$$X = \mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x_l \geq 0 \text{ for all } l = 1, \ldots, L\},$$

Definition 2.3 (Budget set) *The competitive budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ is the set of all feasible consumption bundles x for the consumer who faces market prices p and has wealth w. In this context, "feasible" refers to a bundle that the consumer can afford, meaning its total cost is less or equal to the consumer's wealth.*

- **Preference relation**: Specifies consumer's preference ability across choices, consistently or inconsistently. The preference relation is a binary relation on X, allowing for the comparison of alternatives $x, x' \in X$
  - We can assume strict preferences $x_1 \succ x_2$, if and only if $x_1 \succeq x_2$ holds, and $x_2 \succeq x_1$ does not hold. This means that x_2 can never be as good as x_1. If $x_1 \succeq x_2$ and $x_2 \succeq x_1$, then the relationship is indifference.
  - These relations can be thought of as partitions in the consumption set, given the strict relationships ($x_1 \succ x_2$, $x_2 \succ x_1$, $x_1 \sim x_2$) are mutually exclusive.
- **Utility function representation:** We have a functional representation of the preference relation, where $\mu : X \to \mathbb{R}$ is a utility function, representing $\succeq$ if $x \succeq x'$ implying $u(x) \succeq u(x')$. Recall the **utility function** assigns a number to every possible consumption bundle, such that the more favorable bundle has a higher value.
  - Not all preferences can be represented by a utility function. It is possible, in the case of lexicographical preferences. that the preferences map as a one-to-one function from $\mathbb{R}^{\ltimes} \to \mathbb{R}$ which violates mathematical properties of mapping. Therefore, in some cases, we cannot have a utility function representation.
  - $u(x), ...u(x_n)$ are only admitted by a preference relation where $u(x)$ is complete and transitive *(see below for the axioms of consumer choice).*
  - No $u(x), ...u(x_n)$ are unique. Multiple utility functions can represent the same preferences or choices made by the individual, provided they satisfy certain properties or axioms.

Relation	**Notation**
Weak preference	$x \succeq x'$
Strict preference	$x \succ x'$
Indifference	$x \sim x'$

- **Behavioral assumption:** Identifies the objectives in a consuming choice (i.e. consumer seeks to find and select the alternative that is best in terms of preference or taste).
- **Ordinal utility:** The magnitude of the utility does not matter, it simply indicates how different consumption bundles are ranked. This is what we call ordinal utility.
- **Cardinal utility:** Theoretically, under cardinal utility, the size of the difference (numerically) between two bundles, would hold some type of significance.

- **Taxes and subsidies:** The government always imposes taxes. Taxes are in addition to the price, p_1, p_2 and can be applied in different ways.

  1. **Quantity tax:** The government imposes a per unit tax on a good. For example, Santa Cruz has a sugary beverage tax. For each ounce of a drink, there is a .02 cent tax. A tax introduces a per unit addition to the price, such that $p_1 \Rightarrow (p_1 + t)$.
  2. **Value tax**: A value tax is applied on the price of a good, such that there is a percentage, added additional on the price, in which is a tax. For example, a 7.25% value tax is added to all goods in California. This sales tax is a value tax.
  3. **Lump sum**: The government charges a set fee, regardless of purchases, budget, or income to every person. The opposite of a lump sum is a transfer.
  4. **Subsidy**: A subsidy is the opposite of a tax, in which the government compensates the consumer. For example, a quantity subsidy is a discount per unit of a good purchased, such that prices goes from $p_1 \rightarrow (p_1 - s)$. Subsidies can also be added *ad valorem*, as a value subsidy.

Axioms of consumer choice:

Gives math to the fundamental aspects of consumer behavior/attitudes, the consumer *can* choose; therefore choices are consistent. Properties (axioms) of consumer theory allow us to make certain assumptions about the preferences made, which allow for evaluation of choice implications.

1. **Completeness:** for all x^1 and $x^2 \in X$ either $x^1 \succeq x_2$ or $x^1 \preceq x_2$. This means consumers can actually compare bundles, they discriminate between two goods and make a decision based on the comparison.

2. **Transitivity:** For any three elements x_1, x_2, x_3, if $x^1 \preceq x_2$ and $x^2 \preceq x_3$ then $x^1 \preceq x_3$. This demonstrates that consumer choices are consistent.

3. **Continuity:** for all $x \in \mathbb{R}^{\ltimes}_{+}$ at least as good ($\succeq$) and no better than ($\preceq$) are closed in $\mathbb{R}^{\ltimes}_{+}$. To note, they are closed where the complement is open in the domain. This guarantees that sudden preference reversals cannot occur.

   - Upper and lower contour sets are closed (so the set includes their boundaries)
   - But continuity does not always imply differentiability, although this can be useful. *The case of Leontif preferences where $x^1 \succeq x^2$ if and only if $\min\{x_1^1, x_1^2\} \geq \min\{x_2^1, x_2^2\}$ indicating there is a kink where $x_1 = x_2$*
   - If preferences have a utility representation, then the preferences are homothetic if and only if the utility is homogenous degree one: $u(\alpha * x) = \alpha * u(x)$ for all α

4. **Local nonsatiation:** $\forall x^0 \in \mathbb{R}^n, \forall \epsilon > 0, \exists\ x \in B_\epsilon(x^0) \cap \mathbb{R}^n$ such that $x > x^0$. This rules out any "zones of indifference" or areas where there's always a point within the vicinity of x^o that a consumer may prefer.

5. **Strict monotonicity:** $\forall x^0, x^1 \in \mathbb{R}^n, if x^0 \geq x^1$ then $x^0 \succeq x^1$ while if $x^0 > x^1$ then $x^0 \succ x^1$. This statement means if one bundle contains at least as many of every commodity as the other bundle, it is at least as good. If one bundle contains more of every good, then it is strictly better. This eliminates positively sloping indifference curves (preferred sets must be above, less preferred sets must be below). Monotonicity means MRS is not increasing.

6. **Convexity:** if $x^1 \succeq x^0$ then $tx^1 + (1-t)x^0 \succeq x^0$ for all $t \in [0, 1]$. This assumption can be made strict, where we know $x^1 \neq x^0$ and $x^1 \succeq x^0$ then $tx^1 + (1-t)x^0 \succ x^0$ for all $t \in [0, 1]$. This can be imposed without loss of generality.

   - Consumers prefer a midpoint (some combo of both goods) to only one good.
   - The marginal rate of substitution (MRS) measures the tradeoff between goods, where consumers remain on the indifference curve.
   - The quantities of preference for two goods x^0, x^1 should not depend on the current bundle.

Definition 2.4 (Composite good) *A composite good, often x_2 in our simple model, is a good that represents the composition of all the other goods we may want to buy. We can use the same simple model when we set $p_2 = 1$. This normalizes the price relative to the income, and therefore can be used in the same model.*

Definition 2.5 (Opportunity cost) *The slope of the budget line, $-\frac{p_1}{p_2}$ is the opportunity cost of good 1. This is the relative trade-off between purchasing good 1 and spending your money on the alternative consumption. We have to give up some unit of good 1 in order to have the opportunity to consume good 2 (or the composite good 2).*

Economics has been called the dismal science because it studies the most fundamental of all problems, scarcity. Because of scarcity we all face the dismal reality that there are limits to what we can do. No matter how productive we become, we can never accomplish and enjoy as much as we would like. The only thing we can do without limit is desire more. Because of scarcity, every time we do one thing we necessarily have to forgo doing something else desirable. So there is an opportunity cost to everything we do, and that cost is expressed in terms of the most valuable alternative.[4]

Satiation and Bliss Point. Satiation refers to when a consumer is able to maximize their utility and spend nearly all their income on a bundle of goods. Sometimes a consumer is able to get the *best* bundle of goods, denoted $(\bar{x}_1, \bar{x}_2)$. The closer the consumer's chosen bundle is to $(\bar{x}_1, \bar{x}_2)$, the better off they are; the farther away, the worse off.

Definition 2.6 (satiation or bliss point) *The bundle $(\bar{x}_1, \bar{x}_2)$ is called a **satiation point**. Indifference curves are centered on this point: $(\bar{x}_1, \bar{x}_2)$ is the highest utility level, and bundles farther away lie on lower indifference curves.*

Key topics from the chapters in the book:

Definition 2.7 (Utility) *Utility is not a numeric measure of happiness. Instead, utility is a measure of consumer preference. Utility is a way for us to rank the preferences of a consumer.*

Definition 2.8 (Marginal utility) *The rate of change (or tradeoff) between a consumer's preference for one good over another is called the marginal utility. It is a ration between the rate of change in utility, $\triangle U$ and the rate of change of the good of interest (in this case good 1) - $\triangle x_1$. So we use the following formulas:*

$$MU_1 = \frac{\triangle U}{\triangle x_1} \qquad MU_2 = \frac{\triangle U}{\triangle x_2} \qquad \triangle U = MU_2 \triangle x_2$$

Marginal utility changes with respect to the form of the utility function. Therefore, choice behavior only tells us about rankings of preferences, and not about the marginal utility.

Definition 2.9 (Marginal rate of substitution) *A utility function is used to measure the slope of the indifference curve at a given bundle of goods. MRS therefore is the slope at any given bundle - or the amount a consumer is willing to swap of good 1 for a bit of good 2. We use the formula:*

$$MRS = \frac{\triangle x_2}{\triangle x_1} = -\frac{MU_1}{MU_2}$$

[4]Adapted from Opportunities and Costs, by Dwight Lee. *The Freeman.*

Three main forms of utility functions:

1. **Cobb Douglas**[a] A common utility function with the form:

$$u(x_1, x_2) = x_1^c x_2^d$$

where $c, d > 0$. Cobb Douglas produces well behaved - convex and monotonic - indifference curves.

2. **Perfect complements**[b] Represented by a general utility form:

$$u(x_1, x_2) = min\{\alpha x_1, bx_2\}$$

Where $\alpha, b > 0$. These goods are consumed together in fixed proportions. Therefore, an increase in the bundle of the same proportion of these goods will move utility upward.

3. **Perfect substitutes**[c]

$$u(x_1, x_2) = x_1 + x_2$$

All that matters in a perfect substitute utility is the total number of goods (combined). Simply put a perfect substitute is a function that is constant along each indifference curve and assigns a higher number to more preferred bundles. A more general utility form is

$$u(x_1, x_2) = \alpha x_1 + \beta x_2$$

Where the slope (MRS) is $-\frac{\alpha}{\beta}$. Perfect substitutes are a special case of quasilinear preferences.

[a] See `https://www.econgraphs.org/textbooks/intermediate_micro/scarcity_and_choice/preferences_and_utility/cobb_douglas` for more information and to play around with an actual graph.

[b] See`https://www.econgraphs.org/explanations/consumer/utility/perfect_complements` for more information and to play around with an actual graph

[c] See `https://www.econgraphs.org/explanations/consumer/utility/perfect_substitutes` for more information and to play around with an actual graph.

3 Consumer Choice and Taxes

Chapter 5 (Slide 2,3)

Expanding on consumer choice, we can now say that *optimal choice* is where consumers choose the most preferred bundle from their budget sets.

Definition 3.1 (Optimal choice) *The optimal choice, graphically, is where the budget line (constraint) is tangent to the indifference curve. Analytically, the consumer must maximize utility against a budget, such that*

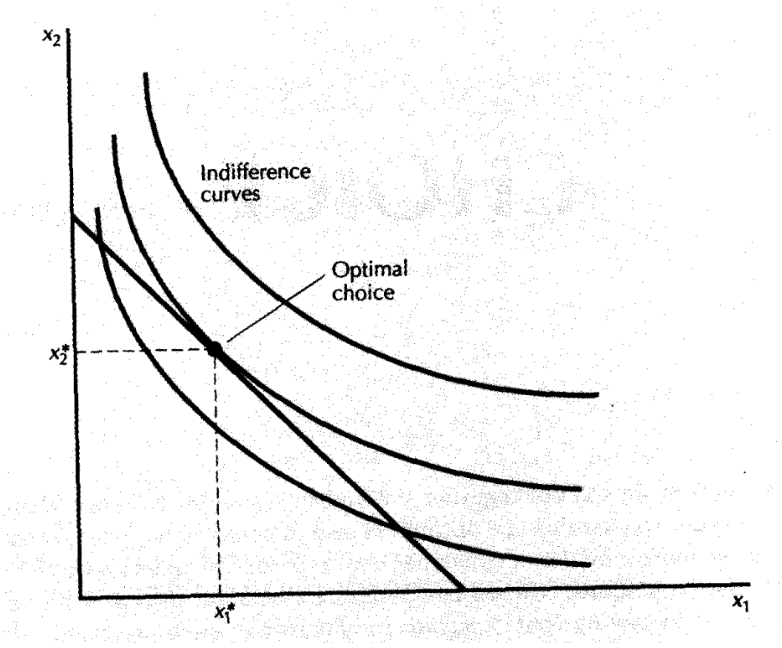
$$\max_{x_1,x_2}\{U(x_1,x_2) \quad s.t.\ p_1x_1+p_2x_2=m\}$$

In order to satisfy the requirements of optimal choice, we must get the following two equalities:

$$\frac{MU_1(x_1^*,x_2^*)}{p_1}=\frac{MU_2(x_1^*,x_2^*)}{p_2} \tag{1}$$

$$p_1x_1^*+p_2x_2^*=m \tag{2}$$

Meeting the optimality conditions both graphically and analytically requires for preferences to be well behaved.[5]



In optimality (and seen in the above graph), we know that the consumers optimal bundle is written, in notation as (x_1^*, x_2^*). This point is where the graphic is indicating *optimal choice.*

Definition 3.2 (Demand function) *The demand function relates to optimal choice, it shows us how much, on aggregate, is demanded at different price levels. The demand function depends both on quantity* (x_1, x_2) *and prices* (p_1, p_2). *Demand functions are written as:*

$$(1) \quad x_1(p_1,p_2,m) \qquad (2) \quad x_2(p_1,p_2,m)$$

[5]Recall that preferences are well-behaved, so that more is preferred to less, we can restrict our attention to bundles of goods that lie on the budget line and not worry about those beneath the budget line.

Mapping the three utility functions to demand functions:

1 **Perfect substitutes:** Because the demand functions now depend both on quantity *and price*, we have three potential cases for the demand function with perfect substitutes. This is *different* than the utility function, where utility depends only on the quantity of both goods. Below we set up three cases which show how price change will effect the relative slopes between the indifference and budget curve:

$$\begin{aligned} &\text{Case 1: } p_1 > p_2 \to && x_1 = 0 \\ &\text{Case 2: } p_1 = p_2 \to && x_1 = [0, \frac{m}{p_1}] \\ &\text{Case 3: } p_1 < p_2 \to && x_1 = \frac{m}{p_1} \end{aligned}$$

In **Case 1**, the price of x_1 is higher. Because they are perfectly substitutable, the consumer only buys x_2 as she gets the same utility from both goods, so she obviously buys the cheaper one. **Case 2** shows its preferred to have a little bit of both, so they buy some amount between 0 and half the income they have. Finally, **Case 3**, shows when p_1 is lower, they buy only good x_1.

2 **Perfect complements:** More simply, the demand function for complements reminds us that we are consuming the same ratio of good 1 and good 2. Because of this, we can just rearrange the budget constraint, such that:

$$\begin{aligned} p_1 x_1 + p_2 x_2 = m \qquad &\text{where} \quad x_1 = x_2 \\ x_1 = x_2 = x \qquad &(p_1 + p_2)x = m \\ \longrightarrow \quad &x^* = \frac{m}{p_1 + p_2} \end{aligned}$$

3 **Cobb Douglas:** Looking at the Cobb-Douglas, we have to solve for the demand curves from the maximization problem. With a standard utility function with C-D preferences, we have: $U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$. The consumer's problem is

$$\max_{x_1, x_2} \; x_1^{\alpha} x_2^{1-\alpha} \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 = m$$

And FOCs end up being:

$$\frac{MU_1}{MU_2} = \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}} = \frac{\alpha}{1-\alpha} \cdot \frac{x_2}{x_1} = \frac{p_1}{p_2} \tag{3}$$

$$\to x_2 = \frac{1-\alpha}{\alpha} \cdot \frac{p_1}{p_2} x_1 \tag{4}$$

$$p_1 x_1 + p_2 \left(\frac{1-\alpha}{\alpha} \cdot \frac{p_1}{p_2} x_1 \right) = m \text{ (from budget constraint)} \tag{5}$$

We end up getting the demand functions for both goods:

$$x_1^*(p_1, p_2, m) = \frac{\alpha m}{p_1}, \qquad x_2^*(p_1, p_2, m) = \frac{(1-\alpha)m}{p_2}$$

Definition 3.3 (Tangency) *Tangency refers to, mathematically, when a line touches another line, but the two lines do not cross. Tangency, therefore, refers to the bundle that has the highest attainable indifference curve that is* just tangent *to the budget line. At this point, the consumer's marginal rate of substitution (MRS) equals the ratio of prices.*[6]

Some facts about the **tangency line**:

- Tangency condition does not have to hold in every case, but the two lines (budget and indifference) will never cross, or else there will be a bundle that is not maximizing utility.
- An indifference curve that is kinked (think $min(x, y)$) will not be tangent to the budget line.
- An indifference curve that intersects the budget only at a boundary (i.e. when either x=0 or y=0) does not meet the tangency condition.
- With convex preferences, the tangency becomes a sufficient condition (this is because a convex indifference curve bends away from the budget line, so there's only one intersection).
- *Takeaway* ⇒ the tangency condition is the a *necessary* condition for an optimal interior solution.

Definition 3.4 (Income tax) *An income tax is a one-off taxation on individuals. It may result in differential welfare effects than a per-unit (quantity) tax. Income tax is non-distortionary, in that the entire budget set of the individual is shrunk, and the relative price of goods does not change. When the income tax is* ***uniform****, it is applied in the sameway across all individuals.*

4 Demand Function and Revealed Preferences

Chapters 6 and 7 (Slide 3)

Next we will look at different types of goods and try to understand how they change with respect to key variables of price and income. These goods, and the relative change in the **demand** for these goods is identified based on economic intuition. Types of goods:

1. **Ordinary**: A good is said to be *ordinary* if its quantity demanded decreases when its own price increases, holding income and other prices constant.

$$\frac{\partial x_1^*(p_1, p_2, m)}{\partial p_1} \leq 0$$

2. **Giffen goods**: A good is said to be *giffen* if its quantity demanded **increases** when its own price increases, holding income and other prices constant. This means the good is valued in a way that values a higher price.

$$\frac{\partial x_1^*(p_1, p_2, m)}{\partial p_1} \geq 0$$

3. **Substitutes**: see previous section.

$$\frac{\partial x_1^*(p_1, p_2, m)}{\partial p_2} \geq 0$$

4. **Complements**: see previous section.

$$\frac{\partial x_1^*(p_1, p_2, m)}{\partial p_2} \leq 0$$

5. **Inferior goods**: As income (m) increases, you actually demand less of the good.

$$\frac{\partial x_1^*(p_1, p_2, m)}{\partial m} \leq 0$$

6. **Normal goods**: As your income increases, you continue to consume more of these goods.

$$\frac{\partial x_1^*(p_1, p_2, m)}{\partial m} \geq 0$$

[6]See the above definition of the optimality conditions where (1) can be rearranged to show you $\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$.

Definition 4.1 (Comparative statistics) *In economics, comparative statics is the analysis of how optimal choices or equilibrium outcomes change when an exogenous parameter changes. It compares two equilibrium states — before and after the change — without modeling the adjustment process between them.*

Formally, if an equilibrium variable x^* depends on a parameter θ, comparative statics examines

$$\frac{\partial x^*}{\partial \theta},$$

to determine how x^* responds to a marginal change in θ.

Example: In consumer theory, comparative statics might study how the optimal demand $x_i^*(p_1, p_2, m)$ changes when income m increases or when the price p_i changes.

5 Appendix

Table of mathematical notation

Symbol(s)	Explanation / Used For	Example
$\varnothing$ (Empty set)	Set with no element	$\lvert\varnothing\rvert = 0$
$\forall$ (For all)	Symbol to indicate all values in a set	$f(x) \geq f(x^*)\ \forall\ x \in \mathbb{R}$
$\mathbb{N}$ (N)	Set of natural numbers	$\forall x, y \in \mathbb{N},\ x + y \in \mathbb{N}$
$\mathbb{Z}$ (Z)	Set of integers	$\mathbb{N} \subseteq \mathbb{Z}$
$\mathbb{Z}_+$ (Z-plus)	Set of positive integers	$3 \in \mathbb{Z}_+$
$\mathbb{Q}$ (Q)	Set of rational numbers	$\sqrt{2} \notin \mathbb{Q}$
$\mathbb{R}$ (R)	Set of real numbers	$\mathbb{R} = (-\infty, \infty)$
$\mathbb{R}_+$ (R-plus)	Set of positive real numbers	$\forall x, y \in \mathbb{R}_+,\ xy \in \mathbb{R}_+$
$\mathbb{C}$ (C)	Set of complex numbers	$\exists z \in \mathbb{C}\ (z^2 + 1 = 0)$
$\mathbb{Z}_n$ (Z-n)	Set of integers modulo n	In $\mathbb{Z}_2$, $1 + 1 = 0$
$\mathbb{R}^3$ (R-three)	Three-dimensional Euclidean space	$(5, 1, 2) \in \mathbb{R}^3$
$f(x), g(x, y), h(z)$	Functions	$f(2) = g(3, 1) + 5$
a_n, b_n, c_n	Sequences	$a_n = \frac{3}{n+2}$
$h, \Delta x$	Limiting variables in derivatives	$\lim_{h\to 0} \frac{e^h - e^0}{h} = 1$
δ, ε	Small quantities in proofs involving limits	$\forall \varepsilon > 0,\ \exists \delta > 0$ such that $\lvert x\rvert < \delta \implies \lvert 2x\rvert < \varepsilon$
$F(x), G(x)$	Antiderivatives	$F(x)' = f(x)$
$\text{dom}(f)$	Domain of f	If $g(x) = \ln x$, then $\text{dom}(g) = \mathbb{R}_+$
$\text{ran}(f)$	Range of f	If $h(y) = \sin y$, then $\text{ran}(h) = [-1, 1]$
$f(x)$	Image of element x under f	$g(5) = g(4) + 3$
$f(X)$	Image of set X under f	$f(A \cap B) \subseteq f(A) \cap f(B)$
$f \circ g$	Composite function	If $g(3) = 5$, $f(5) = 8$, then $(f \circ g)(3) = 8$
$\sum_{i=m}^{n} a_i$	Sum of a_i	$\sum_{i=1}^{5} i^2 = 55$
$\prod_{i=m}^{n} a_i$	Product of a_i	$\prod_{i=1}^{n} i = n!$
$\bar{A}, A^c$	Complement of set A	$\bar{\bar{A}} = A$
$A \cap B$	Intersection of sets	$\{2, 5\} \cap \{1, 3\} = \varnothing$
$A \cup B$	Union of sets	$\mathbb{Z} \cup \mathbb{N} = \mathbb{Z}$
$A \setminus B$	Difference of sets	In general, $A - B \neq B - A$
$A \times B$	Cartesian product of sets	$(11, -35) \in \mathbb{N} \times \mathbb{Z}$
$\mathcal{P}(A)$	Power set of A	$\mathcal{P}(\varnothing) = \{\varnothing\}$
$\lvert A\rvert$	Cardinality of A	$\lvert\mathbb{N}\rvert = \aleph_0$
$\lVert v\rVert$	Norm of vector v	$\lVert(3, 4)\rVert = 5$
$u \cdot v$	Dot product	$u \cdot u = \lVert u\rVert^2$
$u \times v$	Cross product	$u \times u = 0$
$\lim_{n\to\infty} a_n$	Limit of sequence	$\lim_{n\to\infty} \frac{n+3}{2n} = \frac{1}{2}$

Symbol(s)	**Explanation / Used For**	**Example**
$\lim_{x\to c} f(x)$	Limit of function	$\lim_{x\to 3} \frac{\pi \sin x}{2} = \frac{\pi}{2} \lim_{x\to 3} \sin x$
$\sup(A)$	Supremum of A	$\sup([-3,5]) = 5$
$\inf(A)$	Infimum of A	If $B = \{1, \frac{1}{2}, \dots\}$, then $\inf(B) = 0$
$f', f'', f''', f^{(n)}$	Derivatives	$(\sin x)''' = -\cos(x)$
$\int_a^b f(x)\,dx$	Definite integral	$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$
$\int f(x)\,dx$	Indefinite integral	$\int \ln x\,dx = x\ln x - x + C$
$\frac{\partial f}{\partial x}$	Partial derivative	If $f(x,y) = x^2y^3$, then $\frac{\partial f}{\partial x} = 2xy^3$
$a \in A$	a is an element of A	$\frac{2}{3} \in \mathbb{R}$
$a \notin A$	a is not an element of A	$\pi \notin \mathbb{Q}$
$A \subseteq B$	A is a subset of B	$A \cap B \subseteq A$
$A = B$	A is equal to B	If $A = B$, then $A \subseteq B$